A Brief Primer on SIR Models in Economics

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What is a SIR model?

- compartmental epidemiological model
- population of size N passes over time t through three stages (more can be added)
  - **S**usceptible, S_t
  - **I**nfectious, I_t
  - **R**esolved, R_t (recovered or dead)
- $N = S_t + I_t + R_t$ for all t
- initial condition: some infected, $I_0 > 0$; the rest $N - I_0$ susceptible

SIR Equations

- **S** to **I**

$$\frac{dS_t}{dt} = -\beta S_t \frac{I_t}{N}$$

- **I** to **R**

$$\frac{dI_t}{dt} = \beta S_t \frac{I_t}{N} - \gamma I_t$$

- **R** (absorbing state)

$$\frac{dR_t}{dt} = \gamma I_t$$

Key equation

$$\frac{dI_t}{dt} = \beta S_t \frac{I_t}{N} - \gamma I_t \quad \text{(*)}$$

- for a susceptible agent $i \in S_t$, the probability (rate) of being infected is

$$Prob(\mathbf{S} \to \mathbf{I}) = \beta \times \frac{I_t}{N}$$

- β captures
the probability/rate of infection conditional on contact with I person $\times$ **the contact rate**
  - some authors write $\beta = \beta_0 M_t$

- $\frac{I_t}{N}$ is the **probability/rate of contact with I person** (uniform mixing)

Key parameters

- "infectiousness", β
  - tricky, since product of biology (the chance of passing the infection upon meeting **I** person) and behavior/policy (the contact rate)

- "removal rate", γ
  - (mostly) biological – how fast people recover and how many infected die (fraction $\mu\gamma$, approx. 0.3%–0.6% for COVID-19)
  - measured in time^{-1} ($1/\gamma$ is the expected time to recovery/death, e.g., in days)

Dynamics

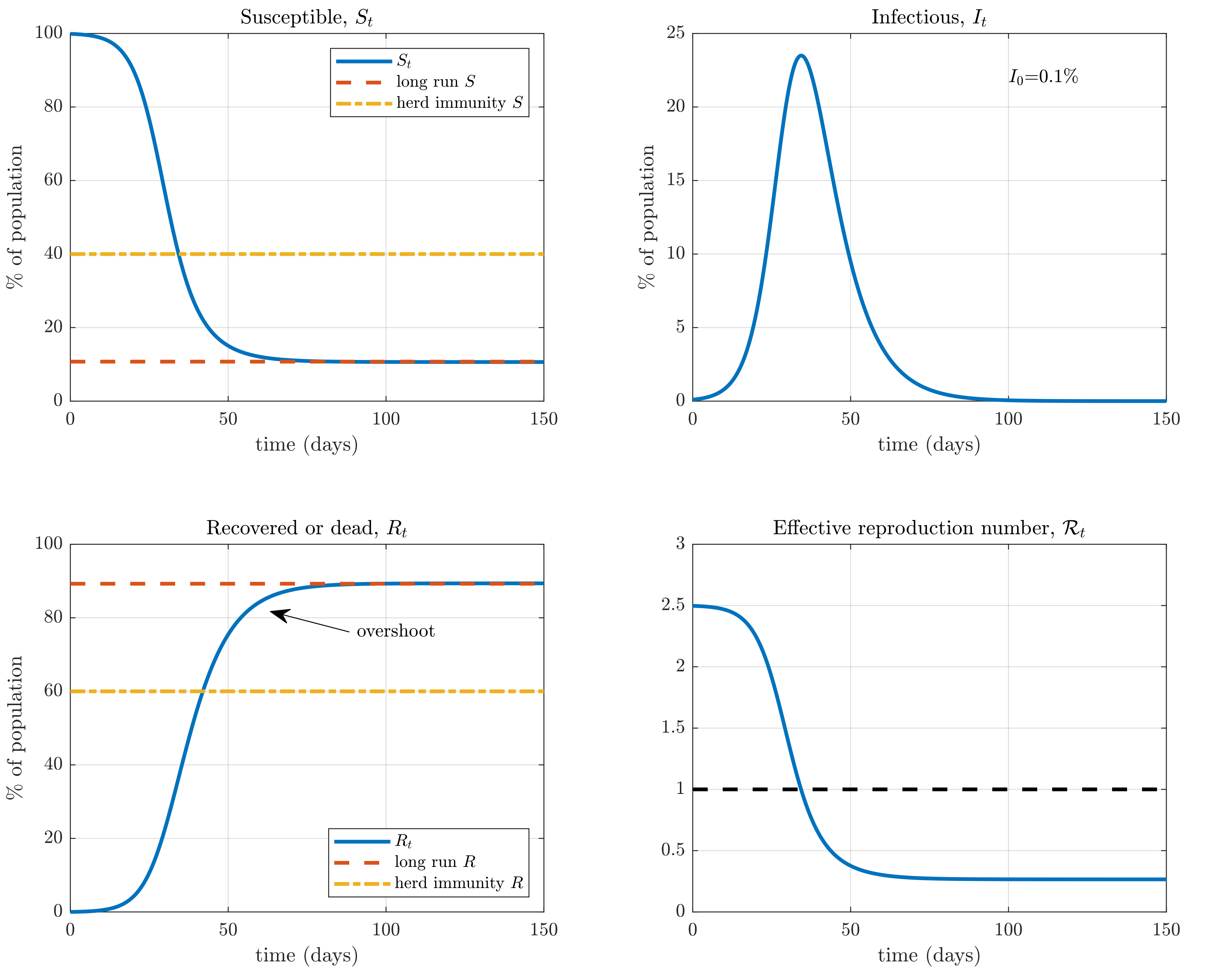
- re-write SIR equation (*) as:

$$\frac{dI_t}{dt} = \beta S_t \frac{I_t}{N} - \gamma I_t = $$
$$= I_t \gamma \left(\frac{\beta}{\gamma}\frac{S_t}{N} - 1\right) \qquad (1)$$

- the current number of infected I_t will grow, $\frac{dI_t}{dt} > 0$ if

$$\frac{\beta}{\gamma}\frac{S_t}{N} > 1 \qquad (2)$$

and decrease otherwise


Susceptible, S_t
S_t
long run S
herd immunity S
% of population
time (days)
Infectious, I_t
I_0=0.1%
% of population
time (days)
Recovered or dead, R_t
overshoot
R_t
long run R
herd immunity R
% of population
time (days)
Effective reproduction number, $\mathcal{R}_t$
time (days)

Basic reproduction number $\mathcal{R}_0$

$\mathcal{R}_0 =$ expected new infections per unit of time generated by the first infected person, when $S_0 \simeq N$

note the confusing notation with R_t (resolved)

- from (1) we see that

$$\mathcal{R}_0 = \frac{\beta}{\gamma}$$

- hence the **importance of $\mathcal{R}_0 = \beta/\gamma$ being larger or smaller than 1**
  - if $\mathcal{R}_0 < 1$ the epidemic never takes off, even at $S_0 = N$, see (2)

- also, note that the early ($S_t \simeq N$) infection growth rate is

$$\frac{dI_t}{dt}/I_t \simeq \beta - \gamma$$

Effective reproduction number

- as the epidemic evolves in time (assuming $\mathcal{R}_0 > 1$) there are less susceptibles
- so, the expected new infections per **I** person and unit of time,

$$\mathcal{R}_t \equiv \frac{\beta}{\gamma}\frac{S_t}{N}$$

decrease over time

- $\mathcal{R}_t$ is called the **effective reproduction number**
- note that if *interventions* (e.g., lockdown, distancing, testing and quarantine) or *behavioral responses* affect β or S_t, I_t they affect $\mathcal{R}_t$ too

Herd immunity

- when $\mathcal{R}_t = 1$ (inflow into state **I** equals outflow from **I**) it is said that "herd immunity" is reached (unstable steady state)
  - herd immunity occurs when
$$\frac{dI_t}{dt} = 0$$
  - that is, when the remaining fraction of susceptibles is
$$\frac{S_t}{N} = \frac{1}{\mathcal{R}_0}$$
  - or, equivalently, when the fraction of recovered or dead (who were infected at some point) is
$$\frac{R_t}{N} = 1 - \frac{1}{\mathcal{R}_0}$$
- Example: if $\mathcal{R}_0 = 2.5$, need **60% of the population** to have been infected to reach herd immunity

Overshoot

- 60% may sound bad enough but the total number of infected, $\int I_t dt$ (and dead) *continues to grow after* herd immunity is reached
- this is known as **overshoot**

Why?

- $\frac{dI_t}{dt} < 0$ after herd immunity is reached means only that there are fewer daily **new** infections (I_t decreases) but $I_t > 0$ still
- important role for isolating known infectious persons

Extensions

- "Exposed" state – incorporate incubation time (around 5 days for COVID-19)
  - SEIR model

- "Quarantine" state – e.g., if an individual tests positive and is isolated from contacts
  - reduces $\frac{I_t}{N}$ in $\beta S_t \frac{I_t}{N}$

- "symptomatic" vs. "asymptomatic" infectious states

SIR model and social networks

- the standard SIR model assumes population-level, uniform mixing
- abstracts from locality and the fact that many infections occur via social contacts (broadly defined)

Network SIR model (NSIR)

- represent the population by graph G (nodes with social links among them)
- probability of a susceptible person i becoming infected at time t depends on i's social contacts

$$Prob(\mathbf{S} \to \mathbf{I}) = \beta \frac{\sum_{j \in C_G(i)} \mathbf{1}_{x_{jt}=\mathbf{I}}}{\# C_G(i)}$$

- where $C_G(i)$ is the set of person i's contacts (other nodes in G)
- $\# C_G(i)$ is node i's degree (number of contacts)

Effective reproduction number in the NSIR model

- note that the graph structure affects the infection dynamics
- in the standard SIR model, the probability of **S** person i meeting an **I** person is **uniform**, $\frac{I_t}{N}$ $\forall i$ ("representative agent model")
- in the network model this probability is **heterogeneous** and depends on i's social contacts

$$\sigma_{it}(G) \equiv \frac{\sum_{j \in C_G(i)} \mathbf{1}_{x_{jt}=\mathsf{I}}}{\#C_G(i)}$$

- can model 'superspreaders', clusters, local outbreaks, etc.

Policy interventions

- essentially, trying to make smaller the term $\beta S_t \frac{I_t}{N}$
- main benefit: "**flatten the curve**" I_t
  - reduce its peak (ICU capacity constraint)
  - reduce the overshoot and total deaths
- cost: can prolong the epidemic duration

Policy interventions

- simplest, reduced-form way
  - *time-dependent* β, e.g., $\beta_t = (1 - l_t)\beta$ (Atkeson, 2020; Moll, 2020)
    - equivalent to reducing the effective reproduction number $\mathcal{R}_t$
  - model "distancing" or "lockdown" as period of lower β_t
- alternative way: reduce the rate of meetings with infectious persons $S_t \frac{I_t}{N}$
  - e.g., $(1 - \lambda)S_t$ meet with $(1 - \lambda)I_t$ where λ is fraction locked down (Alvarez et al. 2020)
  - different meeting technology (Acemoglu et al., 2020)

Testing and quarantine

- introduce state **P** (tested positive) which evolves as

$$\frac{dP_t}{dt} = \theta I_t - \gamma P_t$$

  where θ is the testing rate per unit of time

- Note: the "new cases" reported in the news every day correspond to P_t and **not** to I_t (infectious)

- Quarantine or (self-)isolation:
  - assume **P** agents are removed from meetings
  - or meet on a restricted-contacts graph G_Q

Contact tracing

- key role for the graph G
- tracking, testing and isolating infected social contacts of positive (**P**) agents

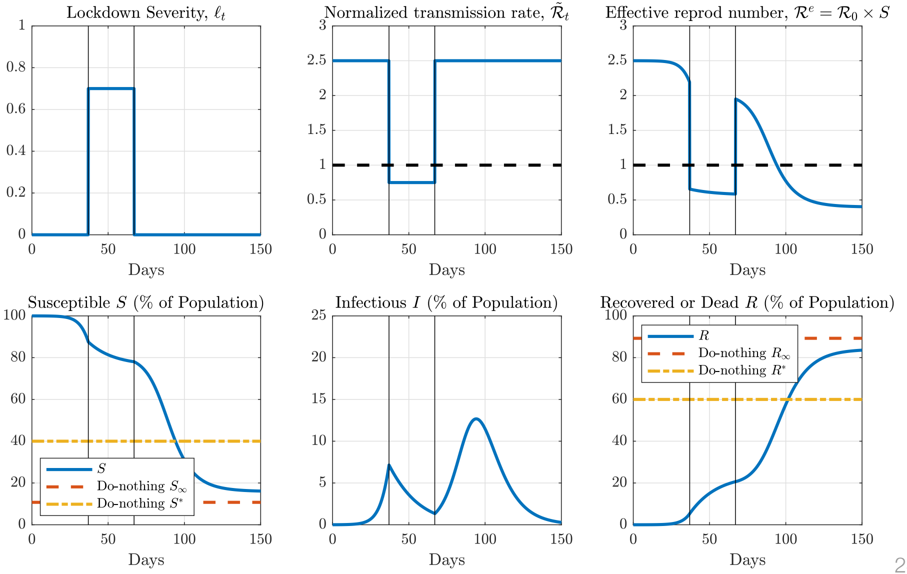
$$Prob(\mathbf{I} \rightarrow \mathbf{P}) = \theta + \phi \sum_{j \in C_G(i)} \mathbf{1}_{x_{jt} = \mathbf{P}}$$

where ϕ is the contact tracing rate

Examples and discussion

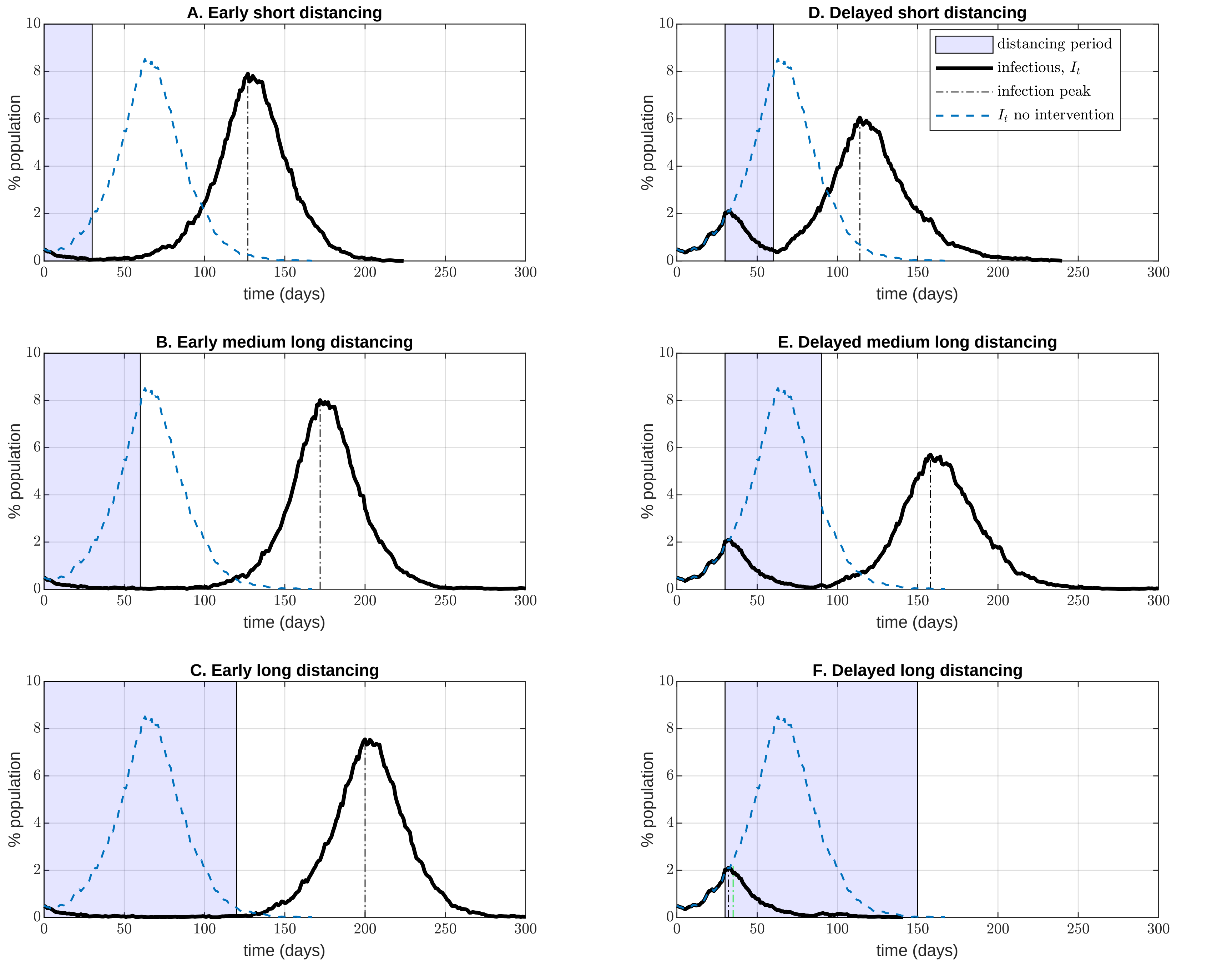
Source: Moll (2020)

Short and Tight Lockdown



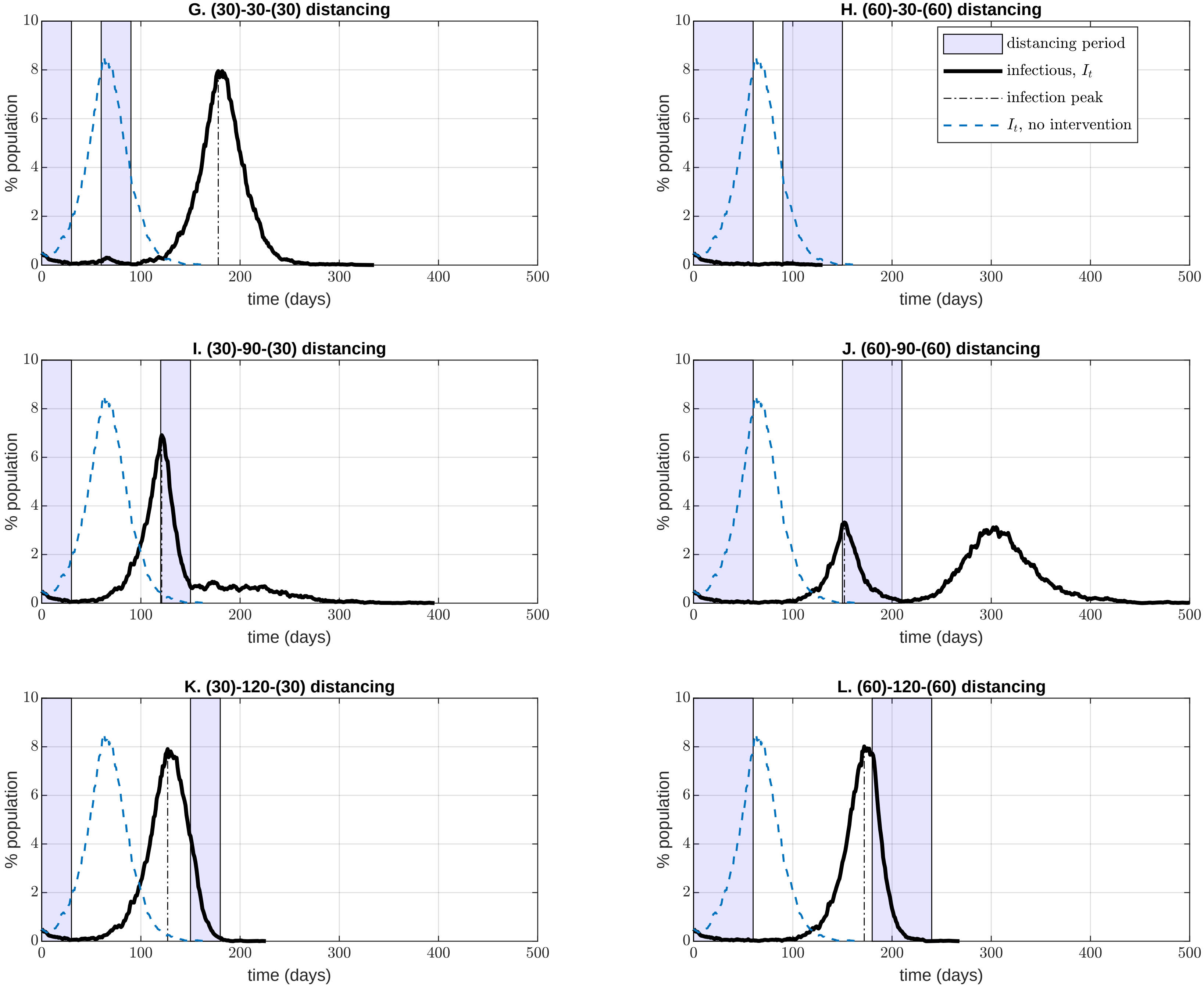
One-off or intermittent distancing

Source: Karaivanov (2020)



Note: assumed initial infection rate 0.5%

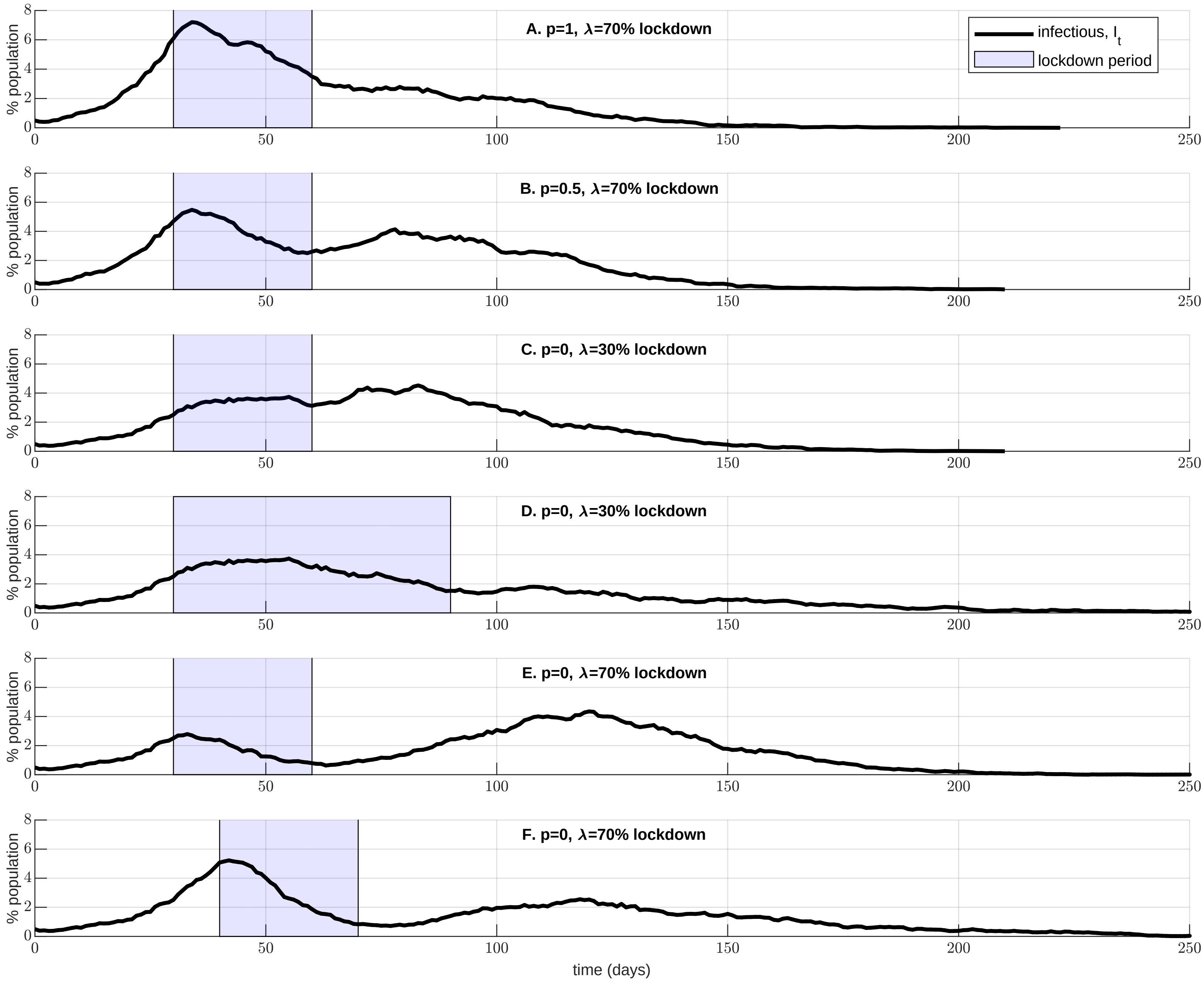
Source: Karaivanov (2020)



Note: assumed initial infection rate 0.5%

Lockdown exit

Source: Karaivanov (2020)



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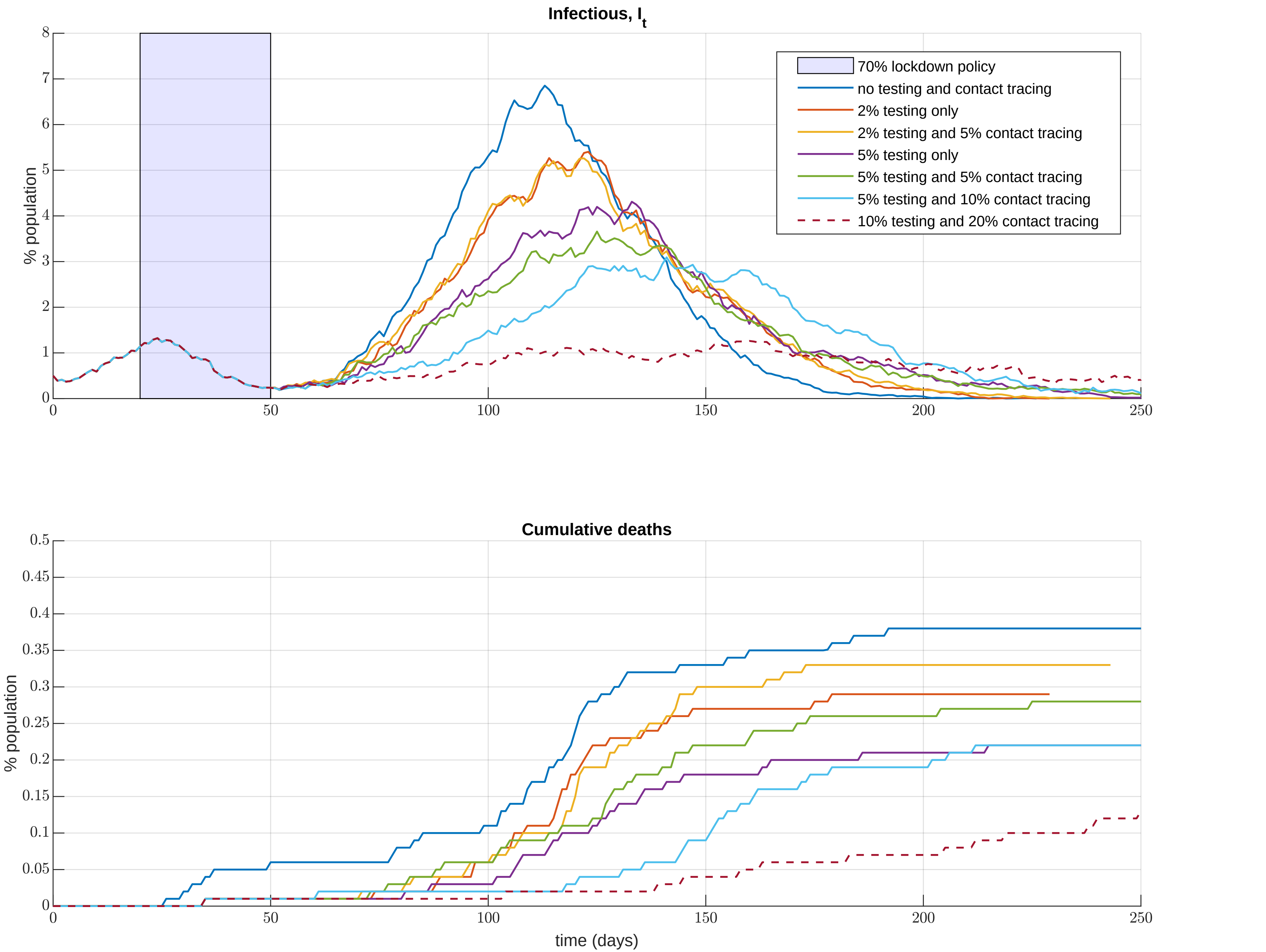
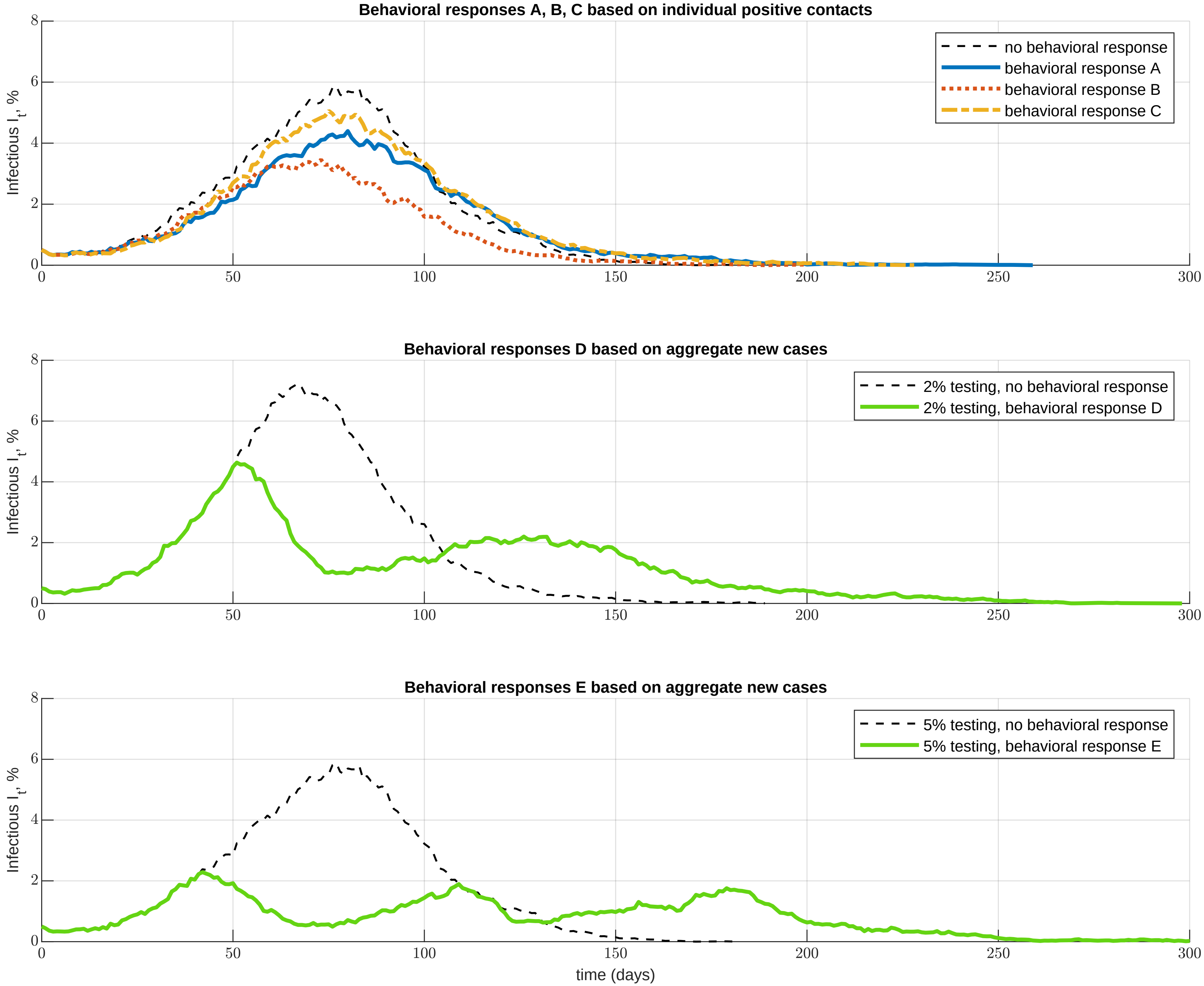


Behavioral responses

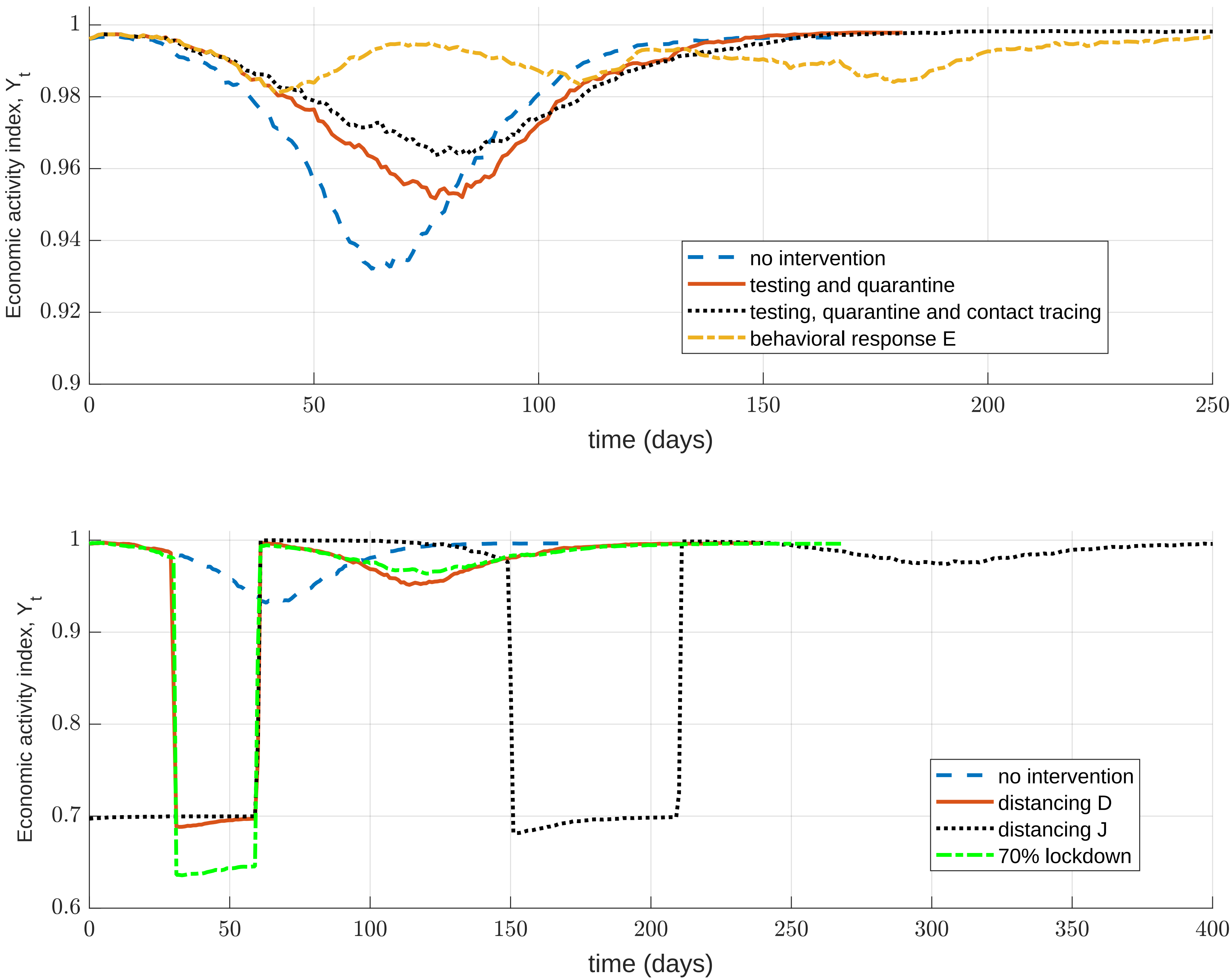
- assume timely information on positive cases is available
- responses (A, B, C) based on individual circumstances – e.g., a contact who tested positive
  - permanent reduction in contacts (upper bound)
  - temporary reduction (while a contact is positive)
- responses D, E based on aggregate data (x new cases in the past y days)

Source: Karaivanov (2020)



Impact on the economy?

- requires (heroic) assumptions about
  - productivity in lockdown and/or when infectious (symptomatic vs. asymptomatic)
  - value of life / years
- current frontier working papers look at interventions and outcomes by segments
  - by industry: e.g., retail vs. IT jobs are affected very differently
  - or, by population cohorts: elderly vs. work-age vs. school-age


Economic activity index, Y_t
time (days)
no intervention
testing and quarantine
testing, quarantine and contact tracing
behavioral response E
Economic activity index, Y_t
time (days)
no intervention
distancing D
distancing J
70% lockdown

Source: Moll (2020)

Some Unpleasant Lockdown Arithmetic

- If lockdowns only option, how long do effective ones need to last?
- Key: need to reach herd immunity. So: how long to reach that?
- Optimistically assume perm immunity, $\mathcal{R}_0 \downarrow$ to 2 (better hygiene...)

$$\text{herd immunity threshold} = 1 - 1/\mathcal{R}_0 = 50\%$$

- Simple back of envelope calculation for U.S. Assumptions:
  1. 10% have had disease $\Rightarrow$ need additional 40% $\approx$ 100 million
  2. lockdown suppresses $\mathcal{R}_t^e$ to 1, infections rolled over (sl 40-41)
     ($\mathcal{R}_t^e = 1$ close to current US estimate)
  3. 200,000 new infections per day (current official count $\approx$ 30,000)
- $\Rightarrow$ need some sort of lockdown / control for

$$\frac{100 \text{ million}}{200,000} = 500 \text{ days}$$

- Note: optimistic calculation assuming low $\mathcal{R}_0$, permanent immunity

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