Midterm Answer Key

Economics 435: Quantitative Methods

Fall 2011

1 Warmup

a) Yes. A proof is not required, but here it is:

$$\begin{aligned}
E(\bar{x}_n) &= E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} E\left(x_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} \left(\mu_E I\left(i \text{ is even}\right) + \mu_O I\left(i \text{ is odd}\right)\right) \\
&= \mu_E \left(\frac{1}{n}\sum_{i=1}^{n} I\left(i \text{ is even}\right)\right) + \mu_O \left(\frac{1}{n}\sum_{i=1}^{n} I\left(i \text{ is odd}\right)\right) \\
&= \mu_E \left(\frac{1}{n}\frac{n}{2}\right) + \mu_O \left(\frac{1}{n}\frac{n}{2}\right) \\
&= \frac{\mu_E + \mu_O}{2}
\end{aligned}$$

b) As usual, $var(\bar{x}_n) = \frac{\sigma^2}{n}$. A proof is not required, but here it is. First, note that since the observations are independent $cov(x_i, x_j) = 0$ for all $i \neq j$. Then:

$$\begin{aligned}
var\left(\bar{x}_n\right) &= var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&= \frac{1}{n^2} var\left(\sum_{i=1}^{n} x_i\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} cov(x_i, x_j)\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} var(x_i)\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} \sigma^2\right) \\
&= \frac{1}{n^2} n\sigma^2 \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

c) Yes. A proof is not required, but here it is. Let $y_j = \frac{x_{2j-1}+x_{2j}}{2}$ and let $N = n/2$. Then:

$$\begin{aligned}
\bar{x}_n &= \frac{1}{n}\sum_{i=1}^{n} x_i \\
&= \frac{1}{2N}\sum_{j=1}^{N} x_{2j-1} + x_{2j} \\
&= \frac{1}{2N}\sum_{j=1}^{N} 2y_j \\
&= \frac{1}{N}\sum_{j=1}^{N} y_j \\
&= \bar{y}_N
\end{aligned}$$

Now, since the x's are independent, so are the y's. In addition, the y's are identically distributed: $y \sim N(\theta, \frac{\sigma^2}{2})$. Therefore we have a random sample of size N on y and can apply the law of large numbers to say that $\bar{y}_N \rightarrow^p \theta$. Since $\bar{x}_n = \bar{y}_N$, this implies $\bar{x}_n \rightarrow^p \theta$ as well.

2 The Mincer human capital model

a) We use our standard procedure

$$\begin{aligned}
\text{plim } \hat{\beta}_1 &= \text{plim } \frac{\hat{cov}(LNWAGE, EDUC)}{\hat{var}(EDUC)} && \text{(usual OLS formula)} \\
&= \frac{\text{plim } \hat{cov}(LNWAGE, EDUC)}{\text{plim } \hat{var}(EDUC)} && \text{(by Slutsky)} \\
&= \frac{cov(LNWAGE, EDUC)}{var(EDUC)} \\
&= \frac{cov(\beta_0 + \beta_1 EDUC + \beta_2 EXPER + u, EDUC)}{var(EDUC)} && \text{(by substitution)} \\
&= \beta_1 + \beta_2 \frac{cov(EXPER, EDUC)}{var(EDUC)} \\
&= \beta_1 + \beta_2 \frac{cov(AGE - EDUC - 6, EDUC)}{var(EDUC)} \\
&= \beta_1 + \beta_2 \frac{cov(AGE, EDUC) - var(EDUC)}{var(EDUC)} \\
&= \beta_1 + \beta_2 \frac{-var(EDUC)}{var(EDUC)} && \text{(since } cov(AGE, EDUC) = 0\text{)} \\
&= \beta_1 - \beta_2
\end{aligned}$$

b) We can substitute $(AGE - EDUC - 6)$ for $EXPER$ in the original model and get:

$$\begin{aligned}
LNWAGE_i &= \beta_0 + \beta_1 EDUC_i + \beta_2 EXPER_i + u_i \\
&= \beta_0 + \beta_1 EDUC_i + \beta_2 (AGE_i - EDUC_i - 6) + u_i \\
&= (\beta_0 - 6\beta_2) + (\beta_1 - \beta_2) EDUC_i + \beta_2 AGE_i + u_i
\end{aligned}$$

Since $E(u|EDUC, AGE) = 0$:

$$\text{plim } \hat{\beta}_1 = \beta_1 - \beta_2$$

c) The coefficient $\hat{\beta}_1$ (and thus its probability limit) doesn't even exist, because the explanatory variables in this regression are perfectly collinear ($EXPER$ is an exact linear function of $EDUC$ and AGE).

d) These regressions would tend to underestimate β_1.

3 Vector autoregressions

a) First we substitute to get:

$$\begin{aligned}
m_t &= b_0 y_t + b_1 y_{t-1} + b_2 m_{t-1} + v_t \\
&= b_0 (a_1 y_{t-1} + a_2 m_{t-1} + u_t) + b_1 y_{t-1} + b_2 m_{t-1} + v_t \\
&= (b_0 a_1 + b_1) y_{t-1} + (b_0 a_2 + b_2) m_{t-1} + (b_0 u_t + v_t) \\
&= \pi_1 y_{t-1} + \pi_2 m_{t-1} + \epsilon_t
\end{aligned}$$

Then we note that:

$$\begin{aligned} E(\epsilon_t|y_{t-1}, m_{t-1}) &= E(b_0 u_t + v_t|y_{t-1}, m_{t-1}) \\ &= b_0 E(u_t|y_{t-1}, m_{t-1}) + E(v_t|y_{t-1}, m_{t-1}) \\ &= b_0 0 + 0 \\ &= 0 \end{aligned}$$

b)

$$\begin{aligned} \sigma_{u\epsilon} &= cov(u_t, \epsilon_t) \\ &= cov(u_t, b_0 u_t + v_t) \\ &= b_0 cov(u_t, u_t) + cov(u_t, v_t) \\ &= b_0 \sigma_u^2 + 0 \\ &= b_0 \sigma_u^2 \end{aligned}$$

c) Note that:

$$\begin{aligned} b_0 &= \frac{\sigma_{u\epsilon}}{\sigma_u^2} \\ b_1 &= \pi_1 - b_0 a_1 \\ b_2 &= \pi_2 - b_0 a_2 \end{aligned}$$

So applying the analog principle produces the estimators

$$\begin{aligned} \hat{b}_0 &= \frac{\hat{\sigma}_{u\epsilon}}{\hat{\sigma}_u^2} \\ \hat{b}_1 &= \hat{\pi}_1 - \frac{\hat{\sigma}_{u\epsilon}}{\hat{\sigma}_u^2} \hat{a}_1 \\ \hat{b}_2 &= \hat{\pi}_2 - \frac{\hat{\sigma}_{u\epsilon}}{\hat{\sigma}_u^2} \hat{a}_2 \end{aligned}$$

d)

$$\begin{aligned}
\text{plim } \hat{b}_0 &= \text{plim } \frac{\hat{\sigma}_{u\epsilon}}{\hat{\sigma}_u^2} \\
&= \frac{\text{plim } \hat{\sigma}_{u\epsilon}}{\text{plim } \hat{\sigma}_u^2} \qquad \text{(Slutsky)} \\
&= \frac{\sigma_{u\epsilon}}{\sigma_u^2} \qquad \text{(given)} \\
&= \frac{b_0\sigma_u^2}{\sigma_u^2} \qquad \text{(previously shown)} \\
&= b_0 \\
\text{plim } \hat{b}_1 &= \text{plim } \left(\hat{\pi}_1 - \frac{\hat{\sigma}_{u\epsilon}}{\hat{\sigma}_u^2}\hat{a}_1\right) \\
&= \text{plim } \hat{\pi}_1 - \text{plim } \frac{\hat{\sigma}_{u\epsilon}}{\hat{\sigma}_u^2}\text{plim } \hat{a}_1 \qquad \text{(Slutsky)} \\
&= \pi_1 - \text{plim } \frac{\hat{\sigma}_{u\epsilon}}{\hat{\sigma}_u^2}a_1 \qquad \text{(given)} \\
&= \pi_1 - b_0a_1 \qquad \text{(previously shown)} \\
&= (b_0a_1 + b_1) - b_0a_1 \\
&= b_1 \\
\text{plim } \hat{b}_2 &= \text{plim } \left(\hat{\pi}_2 - \frac{\sigma_{u\epsilon}}{\sigma_u^2}\hat{a}_2\right) \\
&= \text{plim } \hat{\pi}_2 - \text{plim } \frac{\hat{\sigma}_{u\epsilon}}{\hat{\sigma}_u^2}\text{plim } \hat{a}_2 \qquad \text{(Slutsky)} \\
&= \pi_2 - b_0a_2 \qquad \text{(given/previously shown)} \\
&= (b_0a_2 + b_2) - b_0a_2 \\
&= b_2
\end{aligned}$$