# Midterm Answer Key

Economics 435: Quantitative Methods

Fall 2010

## 1 Cluster samples

**a**) Yes.

$$\begin{aligned}
E(\bar{x}) &= E\left(\frac{1}{nS}\sum_{s=1}^{S}\sum_{i=1}^{n} x_{is}\right) \\
&= \frac{1}{nS}\sum_{s=1}^{S}\sum_{i=1}^{n} E(x_{is}) \\
&= \frac{1}{nS}\sum_{s=1}^{S}\sum_{i=1}^{n} E(\mu + v_s + u_{is}) \\
&= \frac{1}{nS}\sum_{s=1}^{S}\sum_{i=1}^{n} \mu \\
&= \frac{1}{nS}(Sn\mu) \\
&= \mu
\end{aligned}$$

**b**)

$$\begin{aligned}
var(x_{is}) &= var(\mu + v_s + u_{is}) \\
&= var(v_s) + var(u_{is}) + 2cov(v_s, u_{is}) \\
&= \sigma_v^2 + \sigma_u^2 + 0 \\
&= \sigma_v^2 + \sigma_u^2
\end{aligned}$$

**c**) First we need to find the covariance:

$$\begin{aligned}
cov(x_{is}, x_{js}) &= cov(\mu + v_s + u_{is}, \mu + v_s + u_{js}) \\
&= cov(v_s, v_s) + cov(v_s, u_{js}) + cov(u_{is}, v_s) + cov(u_{is}, u_{js}) \\
&= \sigma_v^2 + 0 + 0 + 0 \\
&= \sigma_v^2
\end{aligned}$$

and then we plug into the usual formula for the correlation:

$$\begin{aligned} corr(x_{is}, x_{js}) &= \frac{cov(x_{is}, x_{js})}{\sqrt{var(x_{is})var(x_{js})}} \\ &= \frac{\sigma_v^2}{\sqrt{(\sigma_v^2 + \sigma_u^2)(\sigma_v^2 + \sigma_u^2)}} \\ &= \frac{\sigma_v^2}{\sigma_v^2 + \sigma_u^2} \end{aligned}$$

**d**)

$$\begin{aligned} var(\bar{x}) &= var\left(\frac{1}{nS}\sum_{s=1}^{S}\sum_{i=1}^{n} x_{is}\right) \\ &= var\left(\frac{1}{nS}\sum_{s=1}^{S}\sum_{i=1}^{n} \mu + v_s + u_{is}\right) \\ &= \frac{1}{(nS)^2} var\left(\sum_{s=1}^{S}\sum_{i=1}^{n} \mu + \sum_{s=1}^{S}\sum_{i=1}^{n} v_s + \sum_{s=1}^{S}\sum_{i=1}^{n} u_{is}\right) \\ &= \frac{1}{(nS)^2} var\left(nS\mu + \sum_{s=1}^{S} nv_s + \sum_{s=1}^{S}\sum_{i=1}^{n} u_{is}\right) \\ &= \frac{1}{(nS)^2}\left(var(nS\mu) + \sum_{s=1}^{S} var(nv_s) + \sum_{s=1}^{S}\sum_{i=1}^{n} var(u_{is})\right) \\ &= \frac{1}{(nS)^2}\left(0 + n^2 S\sigma_v^2 + nS\sigma_u^2\right) \\ &= \frac{\sigma_v^2}{S} + \frac{\sigma_u^2}{nS} \end{aligned}$$

**e**) Yes. I didn't ask for proof, but here it is. In a random sample of size $nS$:

$$\begin{aligned} var(\bar{x}) &= \frac{var(x)}{nS} \\ &= \frac{\sigma_v^2 + \sigma_u^2}{nS} \\ &= \frac{\sigma_v^2}{nS} + \frac{\sigma_u^2}{nS} \\ &< \frac{\sigma_v^2}{S} + \frac{\sigma_u^2}{nS} \end{aligned}$$

**f**) Yes. To see this note that:

$$\begin{aligned} \lim_{S\to\infty} var(\bar{x}) &= \lim_{S\to\infty}\left(\frac{\sigma_v^2}{S} + \frac{\sigma_u^2}{nS}\right) \\ &= 0 \end{aligned}$$

This implies that $\bar{x} \to^{m.s.} \mu$, so $\bar{x} \to^{p} \mu$ as well.

**g)** No. To see this note that:

$$\lim_{n\to\infty} var(\bar{x}) = \lim_{n\to\infty}\left(\frac{\sigma_v^2}{S}+\frac{\sigma_u^2}{nS}\right) = \frac{\sigma_v^2}{S} \neq 0$$

## 2 Permanent income and the black-white test score gap

**a)**

$$\begin{aligned} GAP1(C) &= E(s|b=1,c=C) - E(s|b=0,c=C) \\ &= (\gamma_0+\gamma_1+\gamma_2 C) - (\gamma_0+\gamma_2 C) \\ &= \gamma_1 \end{aligned}$$

**b)**

$$\begin{aligned} GAP2(P) &= E(s|b=1,p=P) - E(s|b=0,p=P) \\ &= (\beta_0+\beta_1+\beta_2 P) - (\beta_0+\beta_2 P) \\ &= \beta_1 \end{aligned}$$

**c)** Yes.

**d)**

$$\widehat{GAP1}(C) = \hat{\gamma}_1$$

**e)** Let's try the hint out:

$$\begin{aligned} s &= \beta_0+\beta_1 b+\beta_2 p+u \\ &= \beta_0+\beta_1 b+\beta_2(\delta_0+\delta_1 b+\delta_2 c+w)+u \\ &= (\beta_0+\beta_2\delta_0)+(\beta_1+\beta_2\delta_1)b+\beta_2\delta_2 c+(\beta_2 w+u) \end{aligned}$$

Now $E(\beta_2 w+u|b,c) = \beta_2 E(w|b,c)+E(u|b,c) = \beta_2 * 0+0 = 0$. This means that the model above fits equation (2) where $\gamma_0 = (\beta_0+\beta_2\delta_0)$, $\gamma_1 = (\beta_1+\beta_2\delta_1)$, $\gamma_2 = \beta_2\delta_2$, and $v = (\beta_2 w+u)$. Solving for $\beta_1$ we get:

$$\beta_1 = \gamma_1 - \delta_1\gamma_2/\delta_2$$

This yields the estimator:

$$\begin{aligned} \widehat{GAP2}(P) &= \hat{\beta}_1 \\ &= \hat{\gamma}_1 - \hat{\delta}_1\hat{\gamma}_2/\hat{\delta}_2 \end{aligned}$$

**f)**

$$\begin{aligned} \widehat{GAP1}(C) &= \hat{\gamma}_1 \\ &= -0.426 \\ \widehat{GAP2}(P) &= \hat{\gamma}_1 - \hat{\delta}_1\hat{\gamma}_2/\hat{\delta}_2 \\ &= -0.426-(-0.306*0.205/0.524) \\ &= -0.306 \end{aligned}$$

By the way, you might have noticed that $\widehat{GAP2}(P)$ and $\hat{\delta}_2$ are both -0.306, but that's just a coincidence.

**g)** Yes.