Exam #1

Economics 435: Quantitative Methods

Spring 2006

1 Fitted values and residuals (33 points)

Suppose we are estimating an SLR[1] model. Let $\hat{\beta}_0$ and $\hat{\beta}_1$ be the usual OLS estimators of β_0 and β_1. Let:

$$\begin{aligned} \hat{y}_i &= \hat{\beta}_0 + \hat{\beta}_1 x_i \\ \hat{u}_i &= y_i - \hat{y}_i \end{aligned}$$

a) Is the fitted value $\hat{y}_i$ a consistent estimator of y_i? That is, is it the case[2] that:

$$\text{plim } \hat{y}_i = y_i$$

Provide an argument (ideally, a proof) that your answer is correct.

b) Is the residual $\hat{u}_i$ a consistent estimator of u_i? That is, is it the case that:

$$\text{plim } \hat{u}_i = u_i$$

Provide an argument (ideally, a proof) that your answer is correct.

c) Suppose that you are concerned that the data have heteroskedasticity. You are willing to assume that any heteroskedasticity takes a linear form:

$$E(u^2|x) = \alpha_0 + \alpha_1 x$$

After estimating the original regression (of y on x) and calculating the fitted values, you use OLS to estimate a regression of the squared residual $\hat{u}_i^2$ on x_i. Let the resulting regression coefficients be $\hat{\alpha}_0$ and $\hat{\alpha}_1$. Is it the case that:

$$\text{plim } \hat{\alpha}_1 = \alpha_1$$

Provide an argument (a proof is not necessary here) that your answer is correct.

2 Averages (33 points)

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n on the random variable x. Let $\mu = E(x)$ and $\sigma^2 = var(x)$. Let $a_1, a_2, \ldots, a_n$ be a sequence of n numbers.

[1]In case you didn't write down the definition of the SLR model: Let (y, x, u) be a triplet of random variables such that (SLR1) $y = \beta_0 + \beta_1 x + u$ (SLR2) We have a random sample (y_i, x_i) of size n on the random variables (y, x) (SLR3) $E(u|x) = 0$ (SLR4) There is variation in x_i in the sample.

[2]If you're uncomfortable with the idea of a random variable having another random variable as its probability limit, then just think of an equivalent statement: plim $(\hat{y}_i - y_i) = 0$.

a) Let:

$$W_n = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

Find a condition on the a's that imply that W_n is an unbiased estimator of μ.

b) Find $var(W_n)$ as a function of the a's and of σ^2.

c) For any set of numbers $a_1, a_2, \ldots, a_n$ it is the case that:

$$\frac{(a_1 + a_2 + \cdots + a_n)^2}{n} \leq a_1^2 + a_2^2 + \cdots + a_n^2$$

Use this to show that if W_n is an unbiased estimator of μ, then $var(W_n) \geq var(\bar{X}_n)$, where $\bar{X}_n$ is the usual sample average.

3 Best linear predictors (34 points)

Let y and x be two random variables. The *best linear predictor* (BLP) of y given x is defined as the linear function

$$blp(x) = b_0 + b_1 x$$

that minimizes:

$$E\left((y - b_0 - b_1 x)^2\right)$$

Note that we have not assumed that $E(y|x)$ is linear in x, and Note that the BLP is defined for any pair of random variables, no matter whether $E(y|x)$ is linear or not.

a) Show that:

$$\begin{aligned} b_1 &= cov(x, y)/var(x) \\ b_0 &= E(y) - b_1 E(x) \end{aligned}$$

Hint: to find the minimum of a convex function, take the derivative and set it equal to zero.

b) Let $\hat{\beta}_0$ and $\hat{\beta}_1$ be the coefficients from an OLS regression of y on x from a random sample of (y_i, x_i) of size n. Are $\hat{\beta}_0$ and $\hat{\beta}_1$ consistent estimators of b_0 and b_1? Argue that your answer is correct (ideally, with a proof).

c) Does your result require assumption SLR1 or SLR3?