Exam #1 Answer Key

Economics 435: Quantitative Methods

Spring 2006

1 Fitted values and residuals (33 points)

a) No it isn't.

$$\begin{aligned} \text{plim } \hat{y}_i &= \text{plim } (\hat{\beta}_0 + \hat{\beta}_1 x_i) \\ &= \text{plim } \hat{\beta}_0 + \text{plim } \hat{\beta}_1 \text{plim } x_i \\ &= \beta_0 + \beta_1 x_i \\ &= y_i - u_i \neq y_i \end{aligned}$$

b) Yes it is:

$$\begin{aligned} \text{plim } \hat{u}_i &= \text{plim } (y_i - \hat{y}_i) \\ &= \text{plim } y_i - \text{plim } \hat{y}_i \\ &= y_i - (y_i - u_i) \\ &= u_i \end{aligned}$$

c) Yes. A proof was not required, just some sort of explanation, but here's a proof. Let $z_i = u_i^2$ and $w_i = w_i^2$:

$$\begin{aligned} \hat{\alpha}_1 &= \frac{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})(w_i - \bar{w})}{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2} \\ &= \frac{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})w_i - \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})\bar{w}}{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2} \\ &= \frac{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})w_i}{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2} \\ &= \frac{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})z_i + \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})(w_i - z_i)}{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2} \\ &= \frac{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})(z_i - \bar{z}) + \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})(w_i - z_i)}{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2} \end{aligned}$$

Taking probability limits

$$\begin{aligned} \hat{\alpha}_1 &= \text{plim } \frac{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})(z_i - \bar{z}) + \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})(w_i - z_i)}{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2} \\ &= \frac{\text{plim } \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})(z_i - \bar{z}) + \text{plim } \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})(w_i - z_i)}{\text{plim } \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{\text{plim } \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(z_i-\bar{z}) + \frac{1}{n}\sum_{i=1}^{n}\text{plim } (x_i-\bar{x})\text{plim } (w_i-z_i)}{\text{plim } \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2} \\
&= \frac{\text{plim } \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(z_i-\bar{z}) + \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})0}{\text{plim } \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2} \\
&= \frac{cov(x,z)}{var(x)} \\
&= \alpha_1
\end{aligned}$$

Your answer does not need to be this elaborate!

2 Averages (33 points)

a) The condition is $a_1 + a_2 + \cdots + a_n = 1$.s

b) The answer is:

$$var(W_n) = (a_1^2 + a_2^2 + \cdots + a_n^2)\sigma^2$$

c) If W_n is an unbiased estimator then

$$\frac{(a_1 + a_2 + \cdots + a_n)^2}{n} = \frac{1^2}{n} = \frac{1}{n} \leq a_1^2 + a_2^2 + \cdots + a_n^2$$

Multiply both sides of the above by σ^2 and we get:

$$var(W_n) = (a_1^2 + a_2^2 + \cdots + a_n^2)\sigma^2 \geq \frac{\sigma^2}{n} = var(\bar{X}_n)$$

3 Best linear predictors (34 points)

a)

$$\begin{aligned}
E\left((y - b_0 - b_1 x)^2\right) &= E(y^2 - b_0 y - b_1 xy - b_0 y + b_0^2 + b_0 b_1 x - b_1 xy + b_0 b_1 x + b_1^2 x^2) \\
&= E(y^2) - 2b_0 E(y) - 2b_1 E(xy) + b_0^2 + 2b_0 b_1 E(x) + b_1^2 E(x^2)
\end{aligned}$$

Taking derivatives:

$$\begin{aligned}
\partial E/\partial b_0 &= -2E(y) + 2b_0 + 2b_1 E(x) = 0 \\
\partial E/\partial b_1 &= -2E(xy) + 2b_0 E(x) + 2b_1 E(x^2)
\end{aligned}$$

Solving, we get:

$$\begin{aligned}
b_0 &= E(y) - b_1 E(x) \\
b_1 &= \frac{E(xy) - b_0 E(x)}{E(x^2)} = \frac{E(xy) - E(x)E(y)}{E(x^2) - E(x)^2} = cov(x,y)/var(x)
\end{aligned}$$

b) They are consistent estimators.

$$\text{plim } \hat{\beta}_1 = \text{plim } \frac{c\hat{o}v(x,y)}{v\hat{a}r(x)}$$

$$
\begin{aligned}
&= \frac{\text{plim } c\hat{o}v(x,y)}{\text{plim } v\hat{a}r(x)} \\
&= \frac{cov(x,y)}{var(x)} \\
&= b_1 \\
\text{plim } \hat{\beta}_0 &= \text{plim } (\bar{y} - \hat{\beta}_1 \bar{x}) \\
&= \text{plim } \bar{y} - \text{plim } \hat{\beta}_1 \text{plim } \bar{x} \\
&= E(y) - b_1 E(x) \\
&= b_0
\end{aligned}
$$

c) No, because we never used either in proving consistency.