Exam #2 Answer Key

Economics 435: Quantitative Methods

Fall 2009

1 The fixed effects IV estimator

a) First we note that:

$$\begin{aligned}
\Delta y_{it} &= y_{it} - y_{it-1} \\
&= (a_i + \beta_1 x_{it} + u_{it}) - (a_i + \beta_1 x_{it-1} + u_{it-1}) \\
&= \beta_1 \Delta x_{it} + \Delta u_{it}
\end{aligned}$$

We follow our usual procedure for proving consistency with one explanatory variable:

$$\begin{aligned}
\text{plim } \hat{\beta}_1^{FDIV} &= \frac{cov(\Delta y_{it}, \Delta z_{i,t})}{cov(\Delta x_{it}, \Delta z_{i,t})} && \text{by the law of large numbers} \\
&= \frac{cov(\beta_1 \Delta x_{it} + \Delta u_{it}, \Delta z_{i,t})}{cov(\Delta x_{it}, \Delta z_{i,t})} && \text{by substitution of the result above} \\
&= \frac{\beta_1 cov(\Delta x_{it}, \Delta z_{i,t}) + cov(\Delta u_{it}, \Delta z_{i,t})}{cov(\Delta x_{it}, \Delta z_{i,t})} \\
&= \frac{\beta_1 cov(\Delta x_{it}, \Delta z_{i,t}) + 0}{cov(\Delta x_{it}, \Delta z_{i,t})} && \text{by the strict exogeneity condition (\textbf{??})} \\
&= \beta_1
\end{aligned}$$

b)

$$\begin{aligned}
\text{plim } \hat{\beta}_1^{IV} &= \frac{cov(y_{it}, z_{i,t})}{cov(x_{it}, z_{i,t})} && \text{by the law of large numbers} \\
&= \frac{cov(a_i + \beta_1 x_{it} + u_{it}, z_{i,t})}{cov(x_{it}, z_{i,t})} && \text{by (\textbf{??})} \\
&= \frac{cov(a_i, z_{i,t}) + \beta_1 cov(x_{it}, z_{i,t}) + cov(u_{it}, z_{i,t})}{cov(x_{it}, z_{i,t})} \\
&= \frac{0 + \beta_1 cov(x_{it}, z_{i,t}) + 0}{cov(x_{it}, z_{i,t})} && \text{by (\textbf{??}) and (\textbf{??})} \\
&= \beta_1
\end{aligned}$$

c) We need for changes in a state's gasoline taxes to be unrelated to any state-level changes in the demand for gasoline.

d) We need for cross-state differences in gasoline taxes to be unrelated to any cross-state differences in the demand for gasoline.

2 Partial effects in the logit model

a) There are many different ways to write the PDF, but all involve taking the derivative of the CDF:

$$\begin{aligned}
\lambda(z) &= \frac{d\Lambda(z)}{dz} \\
&= \frac{d\frac{e^z}{1+e^z}}{dz} \\
&= \frac{e^z}{(1+e^z)^2} \qquad \text{OR} \\
&= \frac{e^{-z}}{(1+e^{-z})^2} \qquad \text{OR} \\
&= \frac{1}{(e^z + e^{-z} + 2}
\end{aligned}$$

Any of these 3 answers would be correct, as would several other ways of writing it.

b) I derived this in lecture:

$$\begin{aligned}
PE(x_1, \ldots, x_k) &= \frac{\partial \Pr(y = 1|x_1, \ldots, x_k)}{\partial x_1} \\
&= \lambda(\beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k)\beta_1
\end{aligned}$$

c) The PEA is:

$$\begin{aligned}
PEA &= PE(\bar{x}_1, \ldots, \bar{x}_k) \\
&= \lambda(\beta_0 + \beta_1 \bar{x}_1 + \cdots + \beta_k \bar{x}_k)\beta_1 \qquad (1)
\end{aligned}$$

d) We need the $\tilde{\beta}_0$ that solves:

$$\Lambda(\tilde{\beta}_0 + \beta_1 \tilde{x}_1 + \cdots + \beta_k \tilde{x}_k) = \Lambda(\beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k)$$

Solving we get:

$$\begin{aligned}
\tilde{\beta}_0 &= \beta_0 + \beta_1(x_1 - \tilde{x}_1) + \cdots + \beta_k(x_k - \tilde{x}_k) \\
&= \beta_0 - \beta_1 \bar{x}_1 - \cdots - \beta_k \bar{x}_k
\end{aligned}$$

Since that is a well-defined number, we can satisfy the condition.

e) When $x_i = \bar{x}_i$, then $\tilde{x}_i = 0$. So the PEA is:

$$\begin{aligned}
PEA &= \lambda(\tilde{\beta}_0 + \beta_1 0 + \cdots + \beta_k 0)\beta_1 \\
&= \lambda(\tilde{\beta}_0)\beta_1
\end{aligned}$$

3 Combining samples with population data

a) The first data source provides no information on the correlation between being in prison and completing high school, while the second provides no information on the non-prison population.

b) Let $g_p = \Pr(grad = 1|prison = 1)$. Note that $\overline{grad}$ is a consistent and unbiased estimator of g_p. First we note that:

$$\begin{aligned}
\beta_1 &= \Pr(prison = 1|grad = 1) - \Pr(prison = 1|grad = 0) \\
&= \frac{\Pr(prison = 1 \cap grad = 1)}{\Pr(grad = 1)} - \frac{\Pr(prison = 1 \cap grad = 0)}{\Pr(grad = 0)} \\
&= \frac{\Pr(grad = 1|prison = 1)\Pr(prison = 1)}{\Pr(grad = 1)} - \frac{\Pr(grad = 0|prison = 1)\Pr(prison = 1)}{\Pr(grad = 0)} \\
&= \frac{g_p \mu_p}{\mu_g} - \frac{(1-g_p)\mu_p}{1-\mu_g} \\
&= g_p \frac{\mu_p}{\mu_g(1-\mu_g)} - \frac{\mu_p}{1-\mu_g} \qquad (2)
\end{aligned}$$

By the law of large numbers and Slutsky theorem:

$$\begin{aligned}
\text{plim } \hat{\beta}_1 &= \text{plim } (\overline{grad}) \frac{\mu_p}{\mu_g(1-\mu_g)} - \frac{\mu_p}{1-\mu_g} \\
&= g_p \frac{\mu_p}{\mu_g(1-\mu_g)} - \frac{\mu_p}{1-\mu_g} \\
&= \beta_1
\end{aligned}$$

c) It is unbiased.

$$\begin{aligned}
E(\hat{\beta}_1) &= E\left(\overline{grad}\frac{\mu_p}{\mu_g(1-\mu_g)} - \frac{\mu_p}{1-\mu_g}\right) \\
&= E(\overline{grad})\frac{\mu_p}{\mu_g(1-\mu_g)} - \frac{\mu_p}{1-\mu_g} \\
&= g_p \frac{\mu_p}{\mu_g(1-\mu_g)} - \frac{\mu_p}{1-\mu_g} \\
&= \beta_1
\end{aligned}$$

d) The standard deviation is:

$$\begin{aligned}
\sqrt{var(\hat{\beta}_1)} &= \sqrt{var\left(\overline{grad}\frac{\mu_p}{\mu_g(1-\mu_g)} - \frac{\mu_p}{1-\mu_g}\right)} \\
&= \frac{\mu_p}{\mu_g(1-\mu_g)}\sqrt{var(\overline{grad})} \\
&= \frac{\mu_p}{\mu_g(1-\mu_g)}\sqrt{\frac{g_p(1-g_p)}{n}} \qquad (3)
\end{aligned}$$

e) We can construct a consistent estimate by substitution:

$$\sqrt{v\hat{a}r(\hat{\beta}_1)} = \frac{\mu_p}{\mu_g(1-\mu_g)}\sqrt{\frac{\overline{grad}(1-\overline{grad})}{n}}$$