Problem Set #4 Answer Key

Economics 808: Macroeconomic Theory

Fall 2004

1 The cake-eating problem

a) Bellman's equation is:

$$V(k) = \max_{c\in[0,k]} \{\log c + \beta V(k-c)\}$$

b) If this policy is followed:

$$k_t = \beta^t k_0$$

c) If this policy is followed:

$$c_t = (1-\beta)\,\beta^t k_0$$

d) The value function is calculated by simply substituting the sequence $\{c_t\}$ into the utility function:

$$\begin{aligned} V(k) &= \sum_{t=0}^{\infty} \beta^t \log\left((1-\beta)\,\beta^t k\right) \\ &= \sum_{t=0}^{\infty} \beta^t \left(\log\left((1-\beta)\,\beta^t\right) + \log k\right) \\ &= \sum_{t=0}^{\infty} \beta^t \log\left((1-\beta)\,\beta^t\right) + \sum_{t=0}^{\infty} \beta^t \log k \\ &= constant + \frac{1}{1-\beta}\log k \end{aligned}$$

e) The new policy function $c_1(k)$ is the value of c that solves:

$$c_1(k) = \arg\max_{c\in[0,k]} \left\{\log c + \beta\frac{1}{1-\beta}\log(k-c)\right\}$$

Taking first order conditions, the new policy function is

$$c_1(k) = (1-\beta)k$$

f) Since c_1 and c_0 are identical, we have already found the optimal policy function:

$$c(k) = (1-\beta)k$$

g) First we find V_1 by solving the maximization problem:

$$\begin{aligned} V_1(k) &= \max_{c\in[0,1]} \{\log c + \beta V_0(k-c)\} \\ &= \max_{c\in[0,1]} \{\log c + \beta \log(k-c)\} \end{aligned}$$

The first order conditions are:

$$\frac{1}{c} - \frac{\beta}{k-c} = 0$$

Solving for c, we get:

$$c = \frac{1}{1+\beta}k$$

Then we substitute back in to get V_1:

$$\begin{aligned} V_1(k) &= \log\left(\frac{1}{1+\beta}k\right) + \beta \log\left(\frac{\beta}{1+\beta}k\right) \\ &= \mathit{constant} + (1+\beta)\log k \end{aligned}$$

Since V_1 and V_0 are not identical, we continue. Skipping through the algebra:

$$\begin{aligned} V_2(k) &= \mathit{constant} + (1+\beta+\beta^2)\log k \\ V_3(k) &= \mathit{constant} + (1+\beta+\beta^2+\beta^3)\log k \end{aligned}$$

h) The pattern should be clear at this point:

$$V_i(k) = \mathit{constant} + \left(\sum_{j=0}^{i} \beta^j\right) \log k$$

i) Since $\sum_{j=0}^{i} \beta^j = \frac{1}{1-\beta}$, the limit is:

$$V(k) = \mathit{constant} + \frac{1}{1-\beta}\log k$$

j) Not too surprisingly, the optimal policy under this value function is:

$$c(k) = (1-\beta)k$$

2 Optimal growth with Cobb-Douglas utility

a) The Bellman equation can be written:

$$V(k_t) = \max\left\{\log\left(Ak_t^{\alpha} - k_{t+1}\right) + \beta V(k_{t+1})\right\} \tag{1}$$

or similarly.

b) The way that I've set up the Bellman equation, the necessary conditions are:

$$\begin{aligned} \frac{1}{Ak_t^{\alpha} - k_{t+1}} &= \beta V'(k_{t+1}) \\ V'(k_t) &= \alpha A k_t^{\alpha-1} \frac{1}{Ak_t^{\alpha} - k_{t+1}} \\ \lim_{t\to\infty} \beta^t V'(k_t) k_t &= 0 \end{aligned}$$

c) The Euler equation is:

$$\frac{\beta\alpha A k_{t+1}^{\alpha-1}}{c_{t+1}} = \frac{1}{c_t}$$

d) The steady-state capital, output, and consumption are:

$$\begin{aligned} k_\infty &= (\alpha\beta A)^{\frac{1}{1-\alpha}} \\ y_\infty &= Ak_\infty^{\alpha} = A(\alpha\beta A)^{\frac{\alpha}{1-\alpha}} \\ c_\infty &= y_\infty - k_\infty = A(\alpha\beta A)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta A)^{\frac{1}{1-\alpha}} \end{aligned}$$

e) The steady-state savings rate is:

$$\begin{aligned} s_\infty &= \frac{y_\infty - c_\infty}{y_\infty} \\ &= \frac{(\alpha\beta A)^{\frac{1}{1-\alpha}}}{A(\alpha\beta A)^{\frac{\alpha}{1-\alpha}}} \end{aligned}$$

f) When $\alpha = 1$, the Euler equation becomes:

$$\frac{\beta\alpha A}{c_{t+1}} = \frac{1}{c_t}$$

We can rearrange so that:

$$\frac{c_{t+1}}{c_t} = \beta\alpha A$$

In general, the growth rate of a variable x_t is $\frac{x_{t+1}}{x_t} - 1$ (by definition) or $\log\left(\frac{x_{t+1}}{x_t}\right)$ (a convenient approximation), so the growth rate of consumption is:

$$\beta\alpha A - 1$$

or

$$\log(\beta\alpha A)$$

3 Optimal growth with CRRA utility

a) The Bellman equation can be written:

$$V(k_t) = \max \left\{ \frac{(Ak_t^\alpha - k_{t+1})^{1-\sigma}}{1-\sigma} + \beta V(k_{t+1}) \right\} \tag{2}$$

or similarly.

b) The way that I've set up the Bellman equation, the necessary conditions are:

$$\begin{aligned} (Ak_t^\alpha - k_{t+1})^{-\sigma} &= \beta V'(k_{t+1}) \\ V'(k_t) &= \alpha A k_t^{\alpha-1} (Ak_t^\alpha - k_{t+1})^{-\sigma} \\ \lim_{t\to\infty} \beta^t V'(k_t) k_t &= 0 \end{aligned}$$

c) The Euler equation is:

$$\beta \alpha A k_{t+1}^{\alpha-1} c_{t+1}^{-\sigma} = c_t^{-\sigma}$$

d) The steady state is exactly the same as in the Cobb-Douglas case.

e) Since steady-state output, capital and consumption are the same as in the Cobb-Douglas case, so is the steady state savings rate.

f) The Euler equation can be rearranged so that:

$$\left(\frac{c_{t+1}}{c_t}\right)^\sigma = \alpha\beta A$$

So the growth rate is:

$$g_c = (\alpha\beta A)^{1/\sigma} - 1$$

or

$$g_c = \frac{\log \alpha\beta A}{\sigma}$$

4 Optimal growth with linear utility

a) Because of the production technology, there is a maximum possible level of capital, $k_{max} = A^{\frac{1}{1-\alpha}}$ Since capital cannot be bigger than that number, consumption cannot be bigger than $Ak_{max}^\alpha = A^{\frac{1}{1-\alpha}}$. Since $c_t < A^{\frac{1}{1-\alpha}}$, it must be that:

$$\begin{aligned} \sum \beta^t c_t &< \sum \beta^t A^{\frac{1}{1-\alpha}} \\ &< \frac{A^{\frac{1}{1-\alpha}}}{1-\beta} \end{aligned}$$

In other words, any feasible consumption stream provides finite utility.

b) Let $k*$ be the usual steady state capital stock, i.e., $f'(k^*) = 1/\beta$. Suppose you are deciding whether to consume a particular unit of output or save it for tomorrow. Today, it will give you 1 unit of utility. Tomorrow, it will give you $\beta f'(k_{t+1})$ units of utility. If $\beta f'(k_{t+1}) > 1$, then you will save 100% of output. If you save 100% of your output, then $k_{t+1} = f(k_t)$, so 100% saving is optimal if and only if $\beta f'(f(k_t)) > 1$, or equivalently if $f(k_t) < k^*$. If $\beta f'(k_{t+1}) < 1$, you will consume 100% of your output. If this policy were pursued, then we would have $k_{t+1} = 0$, and $\beta f'(0) = \infty > 1$. So consuming 100% of output can never be optimal. If $\beta f'(k_{t+1}) = 1$, or equivalently if $k_{t+1} = k^*$, you will be indifferent between saving and spending. Any consumption allocation is optimal.

This means that the capital stock will follow the law of motion:

$$k_{t+1} = \min\{f(k_t), k^*\} = \min\left\{k_t^\alpha, (\alpha\beta A)^{\frac{1}{1-\alpha}}\right\}$$

In other words, the planner will choose zero consumption until the steady-state capital level is reached. If the planner can reach steady-state consumption starting in period s, then $c_{s-1} = f(k_{s-1}) - k^*$, and $c_t = f(k^*) - k^* = c^*$ for all $t \geq s$.

Between these three problems, we see how the shape of the flow utility function u affects the behavior of the model. I mentioned in class that Cobb-Douglas utility is a special case (the case that $\sigma = 1$) of CRRA utility, and so is linear utility ($\sigma = 0$). The parameter σ measures a person's preference for smoothing consumption over time. The characteristics of the steady state are unrelated to σ. However, the speed at which the steady state is reached is closely related to σ. When σ is low (linear case), consumers will tolerate low early consumption to get to the steady state quickly. When σ is higher (Cobb-Douglas case), consumers will not tolerate low early consumption, and will reach the steady state more slowly.

c) As I said before, the consumer will choose to save if $\beta f'(k) > 1$, and spend if $\beta f'(k) < 1$. When $\alpha = 1$, the consumer will choose to save if $\beta A > 1$ and spend if $\beta A < 1$.

If $\beta A < 1$, there is a solution to the planner's problem - consume everything right away ($c_0 = Ak_0$).

If $\beta A > 1$, there is no solution to the planner's problem.

You didn't have to prove this, but I will. Suppose that there is a solution $\{c_t\}$ which gives utility level U. We can prove such a solution does not exist by finding a feasible allocation that improves on it. A proposed solution will have the characteristic that there exists some time T such that $c_T > 0$. I propose a new solution $\{\hat{c}_t\}$

$$\hat{c}_t = \begin{cases} c_t & \text{for all } t \notin \{T, T+1\} \\ c_T - 1 & \text{if } t = T \\ c_{T+1} + A & \text{if } t = T+1 \end{cases}$$

The utility from this proposed solution is $\hat{U} = U + \beta^{T+1}A - \beta^T$. Since $\beta A > 1$, $\hat{U} > U$. Therefore U could not have been a solution.