Chapter 3

The representative agent model

3.1 Optimal growth

In this course we're looking at three types of model:

1. Descriptive growth model (Solow model): mechanical, shows the implications of a given fixed savings rate.

2. Optimal growth model (Ramsey model): pick the savings rate that maximizes some social planner's problem.

3. Equilibrium growth model: specify complete economic environment, find equilibrium prices and quantities. Two basic demographies - representative agent (RA) and overlapping generations (OLG).

3.1.1 The optimal growth model in discrete time

Time and demography

Time is discrete.

One representative consumer with infinite lifetime. Social planner directly allocates resources to maximize consumer's lifetime expected utility.

Consumer

The consumer has (expected) utility function:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3.1}$$

where $\beta \in (0,1)$ and the flow utility or one-period utility function is strictly increasing ($u'(c) > 0$) and strictly concave ($u''(c) < 0$). In addition we will impose an Inada-like condition:

$$\lim_{c \to 0} u'(c) = \infty \tag{3.2}$$

The social planner picks a nonnegative sequence (or more generally a stochastic process) for c_t that maximizes expected utility subject to a set of constraints.

The technological constraints are identical to those in the Solow model. Specifically, the economy starts out with k_0 units of capital at time zero. Output is produced using labor and capital with the standard neoclassical production function. Labor and technology are assumed to be constant, and are normalized to one.

$$y_t = F(k_t, L_t) = F(k_t, 1) \tag{3.3}$$

Technology and labor force growth can be accommodated in this model by applying the previous trick of restating everything in terms of "effective labor."

Output can be freely converted to consumption c_t, or capital investment x_t, as needed:

$$c_t + x_t \leq y_t \tag{3.4}$$

Capital accumulation is as in the Solow model:

$$k_{t+1} = (1-\delta)k_t + x_t \tag{3.5}$$

Finally, the capital stock cannot fall below zero.

$$k_t \geq 0 \qquad \forall t \tag{3.6}$$

We could also impose a nonnegativity constraint on investment, but it is convenient in setting up the problem to allow for negative investment (eating capital). Let $f(k) = F(k,1) + (1-\delta)k$. Equations (3.4) and (3.5) above can be combined into a single capital accumulation equation:

$$f(k_t) - c_t - k_{t+1} \geq 0 \tag{3.7}$$

Given these constraints, the consumer or the social planner selects a consumption/savings plan to maximize his utility. Formally, the problem is to pick a sequence (stochastic process) c_t to maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3.8}$$

subject to the constraints:

$$\begin{aligned} f(k_t) - c_t - k_{t+1} &\geq 0 & (3.9)\\ k_t &\geq 0 & (3.10)\\ k_0 &> 0 \quad \text{given} & (3.11) \end{aligned}$$

3.1.2 Establishing existence and other basic properties of a solution

Before going on and doing calculus, we would want to establish that a solution exists, that it is unique, and that it can be found by applying some type of first order conditions.

Suppose that time stopped at some large but finite time T. We know how to solve the problem then, right? First we use those assumptions on u and f to prove that:

- A utility-maximizing consumption sequence exists. Recall that any continuous function defined over a compact set has a well-defined maximum on that set. First we notice that continuity of u implies that we have a continuous objective function, as the sum of n continuous functions is also continuous. Next we need to show that our feasible set is closed and bounded. Closed is easy: $c_t \in [0, f(k_t)]$. To show that it is bounded, we note that by the Inada condition $\lim_{K\to\infty} f'(k) = 0$, so there is a maximum attainable level of capital, which we can call $k^{\max}$ and define by:

  $$k^{\max} = (1-\delta)k^{\max} + f(k^{\max})$$

  So a maximum exists.

- The first order conditions hold at the maximum. Recall that the first order condition is a necessary condition only for interior points in the feasible set. Specifically, for a differentiable function g:

$$(\exists \theta^* \in int(\Theta) : g(\theta^*) \geq g(\theta) \forall \theta \in \Theta) \Rightarrow Dg(\theta^*) = 0$$

  Now, since u is differentiable, our objective function is as well. Next we need to show that c_t is always on the interior of its feasible set. If $c_t = 0$, then marginal utility of one more unit of consumption is infinite, so that can't be a maximum. If c_t is at its maximum level and $t < T$, then $c_{t+1} = 0$. Note: The nonnegativity constraint on k_{T+1} is actually binding, so the first order conditions are going to have to be supplemented with an appropriate complementary slackness condition.

- The usual first order conditions are sufficient. If a strictly concave function f defined on a convex set has a point such that $f'(x) = 0$ then x is the unique global maximizer of f. The u function is strictly concave, so the objective function is also strictly concave.

Once we know this, we can apply the usual steps for optimization subject to equality and inequality constraints. We make a "Lagrangian", take first-order conditions and complementary slackness conditions, and we're set.

In the infinite horizon case, the basic principles work the same. There are two major differences:

- It may be possible to achieve unbounded utility, in which case an optimal plan does not exist. (e.g., there is no maximum to the function $\sum_{i=0}^{\infty} c_t$ for $c_t \in [0, 1]$)

- The complementary slackness condition associated with the binding nonnegativity constraint on k_{T+1} will take on the form of a limiting condition known as the "transversality condition" when there is no "last period."

In addition there will be various technical regularity conditions to worry about. The bible for this sort of thing is Stokey and Lucas, though it's more than a little out of date now.

3.1.3 Solution methods: Dynamic programming

First we will go through a solution method called "dynamic programming." Dynamic programming is used a great deal by macroeconomists.

Suppose that we knew the optimal sequence $\{c_t^*(k_0)\}$ for each possible value k_0 of the initial capital stock. Let:

$$V(k_0) \equiv \sum_{t=0}^{\infty} \beta^t u(c_t^*(k_0)) \tag{3.12}$$

$V(k_0)$ is known as the "value function", it's the value of the objective function from a given starting point, assuming that the optimal plan is followed from that starting point. Now, notice that for this problem V can be defined recursively:

$$\begin{aligned} V(k_t) &\equiv \max_{c_t \in [0, f(k_t)]} \left[u\left(c_t\right) + \beta V\left(f(k_t) - c_t\right)\right] \\ &\equiv \max_{k_{t+1}} \left[u\left(f(k_t) - k_{t+1}\right) + \beta V\left(k_{t+1}\right)\right] \end{aligned} \tag{3.13}$$

Some terminology:

- The recursive formulation is known as a Bellman equation. In general let the original problem be

  $$\begin{aligned} \max_{u_t} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) & \qquad \text{subject to} \\ x_{t+1} = g(x_t, u_t) & \\ x_0 & \qquad \text{given} \end{aligned} \tag{3.14}$$

  Then the associated Bellman equation takes the form:

  $$V(x_t) = \max_{u_t} \{ r(x_t, u_t) + \beta V[g(x_t, u_t)] \}$$

  where r and g are known functions

- x_t is called the state variable, and u_t is called the control variable. They can be vectors. The state variable summarizes all current information relevant to decision making, and the control variable is the decision to be made. Sometimes it is convenient to define the problem so that the control variable is just the next value of the state variable ($u_t = x_{t+1}$)

- The optimal plan can be described by a *policy function*. That is, there exists a function h such that the optimal u_t is given by $u_t = h(x_t)$. Then:
$$V(x_t) = r(x_t, h(x_t)) + \beta V[g(x_t, h(x_t))]$$

Option #1: Recursive methods

One option is to solve recursively.

- Suppose we already know the value function. Finding the policy function is a simple univariate (or multivariate if u_t is a vector) optimization problem.
- Suppose we already know the policy function. Finding the value function is simply a matter of plugging the policy function into the objective function.

This suggests a recursive solution method. Suppose we have a reasonable guess of the value function, which we'll call V_0. Any continuous and bounded function is OK here, but a good guess will often speed up the process. Define:

$$V_1(x_t) = \max_{u_t} \left[r\left(x_t, u_t\right) + \beta V_0\left(g(x_t, u_t)\right) \right] \tag{3.15}$$

Define V_2, V_3, ..., in the same manner. Under some technical conditions which are met in this case, V_i will converge (uniformly) to the correct value function V. This solution method is called "value function iteration."

Alternatively, we could start with an initial guess h_0 for the policy function, then substitute in to calculate:

$$V_0 = \sum_{t=0}^{\infty} \beta^t r(x_t, h(x_t)) \qquad \text{where } x_{t+1} = g(x_t, h(x_t)) \tag{3.16}$$

Then we can calculate an improved policy function h_1:

$$h_1(x) = \arg\max_u \left[r(x, u) + V_0(g(x, u)) \right] \tag{3.17}$$

If we repeat this, h_i will converge (uniformly) to the true policy function and V_i will converge to the true value function. This method is called "policy function iteration." In practice, policy function iteration is often faster than value function iteration.

These recursive solution techniques are very powerful tools, but:

- All of this iteration takes a lot of time to do by hand.
- In general, neither the value function nor the policy function will have a closed form.
- As a result, we will usually need to impose specific parametric functional forms (e.g., Cobb-Douglas), choose specific parameter values, and solve numerically on a computer. Now a computer needs a finite representation for all objects it works with. As a result the value function must be approximated using one of the following methods, described in detail by Sargent.
  - Discretization of the state space. Suppose we define an n-element set K such that $k_0 \in K$, and restrict the social planner's choice set to $k_{t+1} \in K$. Then V_i and h_i can each be represented by an n-vector.
  - Approximation of the value function and/or policy function by an nth-order polynomial.

  Once the finite representation is chosen, it is relatively simple to program a computer to pick n real numbers to maximize some well-behaved continuous function.

We will not have computer-based dynamic programming assignments, as the overhead is too high.

Option #2: Calculus/Euler equations

Another solution method is to apply calculus. Under certain conditions, the value function is differentiable, and a solution must satisfy a first order condition as well as a "transversality condition:"

$$\begin{aligned} \frac{\partial r}{\partial u}(x_t, u_t) + \beta V'[g(x_t, u_t)]\frac{\partial g}{\partial u}[x_t, u_t] &= 0 \qquad \text{(FOC)} \\ \lim_{t \to \infty} \beta^t V'(x_t) x_t &= 0 \qquad \text{(TVC)} \end{aligned} \tag{3.18}$$

The transversality condition is closely related to the limit of the complementary slackness condition in the case of finite T. We will ignore it most of the time, but occasionally it's very important. In addition, we can apply a result sometimes called the Benveniste-Scheinkman theorem:

$$V'(x_t) = \frac{\partial r}{\partial x}(x_t, u_t) + \beta V'[g(x_t, u_t)]\frac{\partial g}{\partial x}(x_t, u_t) \tag{3.19}$$

This is kind of a dynamic programming version of the envelope theorem; notice that we can ignore the effect of changes in the state x on the optimal policy u.

Sargent notes that solving a Bellman equation with calculus is a lot easier if we can rewrite it so that $\frac{\partial g}{\partial x} = 0$. For example, if we have defined our Bellman equation so that our control variable is just x_{t+1}, then $g(x_t, u_t) = u_t$ and $\frac{\partial g}{\partial x} = 0$.

Let's apply these results to our model. Define:

$$V(k_t) = \max_{k_{t+1}} \{u(f(k_t) - k_{t+1}) + \beta V(k_{t+1})\} \tag{3.20}$$

By the first order condition we have:

$$\begin{aligned} -u'(f(k_t) - k_{t+1}) + \beta V'(k_{t+1}) &= 0 \\ u'(c_t) &= \beta V'(k_{t+1}) \end{aligned} \tag{3.21}$$

and by the Benveniste-Scheinkman theorem we have:

$$\begin{aligned} V'(k_t) &= u'(f(k_t) - k_{t+1})f'(k_t) \\ &= u'(c_t)f'(k_t) \end{aligned} \tag{3.22}$$

Notice that we have a simple closed form solution for V', but this doesn't mean we know V. Substituting we get something which is known as the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})f'(k_{t+1}) \tag{3.23}$$

The Euler equation has an economic interpretation: along an optimal path the marginal utility from consumption at any point in time is equal to its opportunity cost. The Euler equation is a necessary condition for an optimal policy. Another necessary condition is the transversality condition, which here means:

$$\lim_{t\to\infty} \beta^t u'(c_t)f'(k_t)k_t = 0 \tag{3.24}$$

The transversality condition basically means here that the present value of future capital k_t must be going to zero.

3.1.4 Dynamics

The Euler equation and the capital accumulation equation represent a second order difference equation:

$$\beta u'(c_{t+1})f'(k_{t+1}) = u'(c_t) \tag{3.25}$$

$$k_{t+1} = f(k_t) - c_t \tag{3.26}$$

In order to have a unique solution, a second-order difference equation must be supplemented with exactly two boundary conditions. The first boundary condition is the initial value k_0. If we knew the initial value c_0, then the rest of the values would simply be a matter of plugging into the equations. The second boundary condition will be the transversality condition. It will turn out that there's only one value of c_0 that doesn't violate the TVC.

First, let's find a steady state. A steady state is a pair (c_∞, k_∞) such that:

$$\beta u'(c_\infty)f'(k_\infty) = u'(c_\infty) \tag{3.27}$$

$$k_\infty = f(k_\infty) - c_\infty \tag{3.28}$$

Rearranging (3.27), we get

$$f'(k_\infty) = \beta^{-1} \tag{3.29}$$

Since $f'(0) = \infty$, $f'(\infty) \leq 1$, and $\beta^{-1} > 1$, we know that there is exactly one k_∞ that solves equation (3.27). To find c_∞, just substitute that k_∞ back into equation (3.28). The economy also has a steady state at $k_t = 0$.

Now let's draw a phase diagram (graphics not available here, so this description may not be very useful). Since this is a second order system it will be more complex than the phase diagram for the Solow model. Put k_t on the x axis and c_t on the y axis. Follow these steps:

1. Plot the equation $c_t = f(k_t) - k_t$. Notice that it is shaped like an inverted U. This set of points defines for each level of k_t the value of c_t at which k_{t+1} will equal k_t. If c_t is above this line, then capital is shrinking $k_{t+1} < k_t$, and if c_t is below the line then capital is growing.

2. Plot the equation $f'(k_t) = 1/\beta$. Notice that this is the equation defining the steady-state value of the capital stock. Note that if $k_{t+1} < k_\infty$, then the Euler equation implies $u'(c_t) > u'(c_{t+1})$ which implies $c_{t+1} > c_t$, or that consumption is growing. We'll temporarily blur the distinction

between k_t and k_{t+1}; this is not too unreasonable if the length of a period is short enough.

3. Now, notice that we can divide the graph into quadrants. If (c_t, k_t) ever enters the upper left quadrant, we will get k_t going negative at some point. If it ever enters the lower right quadrant we will get c_t going to zero, which will violate the transversality condition. So the system always has to stay in either the lower left or upper right quadrants. It turns out that there is a unique path going through those quadrants that doesn't end up going off into the bad quadrants, and that defines a unique optimal policy function.

3.1.5 The optimal growth model in continuous time

Time and demography

Time is continuous. One consumer, infinitely lived.

Consumers

The social planner maximizes lifetime utility:

$$U = \int_0^\infty e^{-\rho t} u(c_t) dt \tag{3.30}$$

subject to the constraints

$$\begin{aligned} \dot{k}_t &= y_t - \delta k_t - c_t && (3.31) \\ y_t &= F(k_t, 1) && (3.32) \\ k_t &\geq 0 && (3.33) \\ k_0 &> 0 \qquad \text{given} && (3.34) \end{aligned}$$

Let $f(k) = F(k, 1) - \delta k$. Then we can rewrite the constraints (3.31) and (3.32) as a single constraint:

$$\dot{k}_t = f(k_t) - c_t \tag{3.35}$$

In this model k_t is the "state" variable and c_t is the "control" variable. In continuous time the distinction is clearer - the time path of the state variable must be differentiable. In contrast, the control variable can change discontinuously. The control variable is sometimes called the "jumping" variable.

Solving the model

Here's our solution method. Construct a Hamiltonian (similar to a Lagrangean):

$$H(t) = e^{-\rho t}u(c_t) + \lambda_t(f(k_t) - c_t) \tag{3.36}$$

λ_t is called the *costate* variable. Notice that λ_t is being multiplied by the time derivative of k_t. The solution to a Hamiltonian can be characterized by two first order conditions:

$$\frac{\partial H}{\partial c_t} = 0 \tag{3.37}$$

$$\frac{\partial H}{\partial k_t} + \dot{\lambda}_t = 0 \tag{3.38}$$

and the transversality condition:

$$\lim_{t\to\infty} \lambda_t k_t = 0 \tag{3.39}$$

In this case:

$$\lambda_t = e^{-\rho t}u'(c_t) \tag{3.40}$$

$$\lambda_t f'(k_t) = -\dot{\lambda}_t \tag{3.41}$$

Let's find what $\dot{\lambda}_t$ is:

$$\begin{aligned}\dot{\lambda}_t &= \frac{\partial}{\partial t}\left[e^{-\rho t}u'(c_t)\right] && (3.42)\\ &= -\rho e^{-\rho t}u'(c_t) + e^{-\rho t}u''(c_t)\dot{c}_t && (3.43)\end{aligned}$$

In order to make this tractable, use the CRRA form of the utility function:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \tag{3.44}$$

so that $u'(c) = c^{-\sigma}$ and $u''(c) = -\sigma c^{-\sigma-1}$.

$$\begin{aligned}\dot{\lambda}_t &= -\rho e^{-\rho t}c_t^{-\sigma} + e^{-\rho t}(-\sigma)c_t^{-\sigma-1}\dot{c}_t && (3.45)\\ &= -e^{-\rho t}c_t^{-\sigma}\left[\rho + \sigma\frac{\dot{c}_t}{c_t}\right] && (3.46)\\ &= -\lambda_t\left[\rho + \sigma\frac{\dot{c}_t}{c_t}\right] && (3.47)\end{aligned}$$

Substituting back in and dividing by λ_t:

$$f'(k_t) = \rho + \sigma \frac{\dot{c}_t}{c_t} \tag{3.48}$$

For any x, the quantity $\frac{\dot{x}}{x}$ is its growth rate. So the growth rate of consumption is:

$$\frac{\dot{c}_t}{c_t} = \frac{f'(k_t) - \rho}{\sigma} \tag{3.49}$$

This is the Euler equation for this problem.

Dynamics

Equations (3.49) and (3.35) represent a second-order differential equation. Again we need two boundary conditions. Initial capital is one, the TVC will be the other. We find a steady state in a continuous time model by noting that nothing is growing in a steady state. Since consumption doesn't grow:

$$f'(k_\infty) = \rho \tag{3.50}$$

and steady state consumption is simply $f(k_\infty)$. Compare this to the results in the discrete time case.

Here's a phase diagram.

As you can see, the continuous time and discrete time models are analogous - no substantive differences. Rules for translating:

- If you see x_{t+1} in the discrete time model, subtract x_t, and replace $x_{t+1} - x_t$ with $\dot{x}_t$.
- In continuous time models, try to find the growth rate of each variable.

In general, continuous time is more useful for models of long-run growth, where discrete time is more useful for business cycle models. But mostly it's a matter of taste. I find discrete time to be easier, Romer uses mostly continuous. You have to know both.

3.2 Equilibrium growth

3.2.1 What have we learned so far

The basic optimal growth model on its own doesn't teach us much. Notice that virtually any policy implication of the model follows from the much simpler Solow model. This is because the only thing we've really added is that we have imposed an optimal savings rate which is dependent on the current capital level. We have not shown any relationship between the level of aggregate savings chosen by the social planner and the savings pattern in a market economy.

So why did we do it? The next thing that we have on the agenda is an equilibrium growth model, which will make positive predictions about savings rates, as well as prices and interest rates, in terms of economic fundamentals. This is what we really care about. What will we use?

- Consumers will solve an intertemporal optimization problem. We now know how to work with such problems.
- By solving the normative model and comparing the results to the positive model, we can identify sources and implications of any market failure in the positive model.
- Finding an optimum is generally easier than finding an equilibrium. The baseline model we analyze will satisfy the Arrow-Debreu conditions for Pareto optimality of equilibrium. Once we know this, we can solve for quantities by solving the planner's problem, without having to solve the equilibrium problem .
- Even in cases where the equilibrium model exhibits market failure, the market failure often takes a form such that the equilibrium quantities solve some pseudo-planner's problem.

3.2.2 The model

Expectations

The economy that we will discuss here has two important properties - first, that agents in the model optimize, and second that they care about the future. If this is true, then they need to forecast the future. There are a few different approaches to modeling the forecasts of economic agents:

1. Naive expectations - people expect tomorrow to be just like today.
2. Adaptive expectations - people use some mechanical function of past events to predict future events.
3. More sophisticated adaptive expectations - adaptive expectations, except people change the function. Usually follows some evolutionary approach where new prediction rules arise and are kept as long as they work.
4. Rational expectations - agents *know* the true probability distribution of all future events. An implication of this is that they know exactly how the economy works. This is the most common assumption in modern macroeconomics, ever since the pioneering work of Robert Lucas, Tom Sargent, Edward Prescott, Neil Wallace, et al.
5. Perfect foresight - agents know exactly what will happen in the future. Really a special case of rational expectations where the probability distribution of future events is degenerate (all events have probability one or probability zero).

In this class we will use rational expectations.

In this model, agents will have perfect foresight.

Demography

One consumer with infinite lifetime.

One firm each period.

Consumer

The consumer maximizes lifetime utility:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3.51}$$

subject to the following budget constraint.

The consumer is endowed with one unit of labor, and k_0 units of capital. The consumer supplies labor inelastically at wage w_t and rents capital at rental rate r_t. The consumer also owns the firm and receives profit π_t. This income is spent on either consumption (c_t), new capital (k_{t+1}), or bonds. Bonds are denominated in units of output - one unit of output buys a bond which pays off R_{t+1} units the next period. Putting all of this together, the consumer has the following budget constraint each period:

$$c_t + x_t + b_{t+1} \leq w_t + r_t k_t + R_t b_t + \pi_t \tag{3.52}$$

and the following capital accumulation rule:

$$k_{t+1} = (1 - \delta)k_t + x_t \tag{3.53}$$

The consumer cannot hold negative amounts of capital:

$$k_t \geq 0 \tag{3.54}$$

but can have either positive or negative holdings of bonds. In other words, the consumer can borrow at the going rate.

So what is the solution to this maximization problem? As described, the solution is simple and requires no first order conditions. The solution is to borrow an arbitrarily large amount of money today, then borrow even more tomorrow to pay it back. Since the time horizon is infinite, there will never be a day when these debts must be settled.

An aside: borrowing an ever-increasing amount of money in order to pay off the previous lenders is called a pyramid scheme or a Ponzi game. Charles Ponzi was a Boston businessman of the 19th century that created an investment fund that paid an astonishing rate of return. The way he did this was that the rate of return drew more and more investors, and Ponzi used the new investors funds to pay off the old investors. In addition, because the rate of return was so high, most investors would reinvest. Eventually, of course, he was caught.

Obviously we need a No-Ponzi Game condition. But we don't want to forbid borrowing, so we look for the weakest condition that will rule out a Ponzi game. One solution is to pick any finite number and say that the amount of debt can't go above that number. Define q_t as the time zero value of one unit of consumption at time t:

$$q_t = \prod_{i=1}^{t} (R_i)^{-1} \tag{3.55}$$

In other words, if we give up q_t units of consumption at time zero, we get one unit of consumption at time t. A reasonable condition is that the present value of future debts must be going to zero.

$$\lim_{t\to\infty} q_t b_t = 0 \tag{3.56}$$

Alternatively, the amount of debt must grow more slowly than the rate of interest.

Firms

There is a single, competitive firm. Each period the firm rents labor L_t and capital k_t on the market, and produces "output" according to the production function $F(k_t, L_t)$. The production function has the standard neoclassical properties, and again, we define $f(k) = F(k, 1) + (1 - \delta)k$

Output can take the form of consumption c_t or new capital k_{t+1}. The firm's profit is:

$$c_t + k_{t+1} - w_t L_t - r_t K_t \tag{3.57}$$

Therefore the firm's maximization problem is:

$$\max_{L_t, k_t} F(k_t, L_t) - w_t L_t - r_t K_t \tag{3.58}$$

Notice that the firm's problem is static. This is very much intentional- this kind of model is easiest if you assign property rights so that only one agent has a dynamic maximization problem.

Equilibrium

Let's put together all of the pieces, and define an equilibrium in this model. When setting up a dynamic macroeconomic model, it's very important to define equilibrium precisely. Here's the language:

An equilibrium in this economy is a set of prices $\{p_{kt}\}$, $\{R_t\}$, $\{w_t\}$, $\{r_t\}$, and quantities $\{c_t\}$, $\{k_t\}$, $\{L_t\}$, and $\{b_t\}$ such that:

1. Taking prices and firm profits as given, the quantities solve the consumer's maximization problem.

2. Taking prices as given, the quantities solve the firm's maximization problem.

3. For all t, $L_t = 1$ and $b_t = 0$.

A competitive equilibrium always has two key elements - maximization by price taking agents, and market clearing. In a competitive model, the agents don't worry about the feasibility of their decisions, but the prices adjust until their decisions are feasible. For example, the consumer is allowed to pick any amount of bond holding he wishes, but the price will have to adjust so that the net supply of bonds is zero - there is a lender for every borrower. (3) is our main market clearing condition, but we have also implicitly made another market clearing condition by our notation - the capital selected by the firm and by the consumer are given the same name. Implicitly we are assuming that they pick the same value.

3.2.3 Solving the model

The consumer

First, let's solve the consumer's problem. The Lagrangian is:

$$\begin{aligned} L &= \sum_{t=0}^{\infty} \beta^t u(c_t) && (3.59) \\ &+ \lambda_t (r_t k_t + w_t + R_t b_t - c_t - x_t - b_{t+1}) \\ &+ \theta_t ((1-\delta) k_t + x_t - k_{t+1}) \end{aligned}$$

The first order conditions are:

$$\frac{\partial L}{\partial c_t} = \beta^t u'(c_t) - \lambda_t = 0 \quad (3.60)$$

$$\frac{\partial L}{\partial b_{t+1}} = R_{t+1} \lambda_{t+1} - \lambda_t = 0 \quad (3.61)$$

$$\frac{\partial L}{\partial x_t} = -\lambda_t + \theta_t = 0 \quad (3.62)$$

$$\frac{\partial L}{\partial k_{t+1}} = \lambda_{t+1} r_{t+1} + \theta_{t+1}(1-\delta) - \theta_t = 0 \tag{3.63}$$

$$\tag{3.64}$$

plus the budget constraint (3.52) and (3.53). The transversality conditions are:

$$\lim_{t\to\infty} \theta_t k_t = 0 \tag{3.65}$$

$$\lim_{t\to\infty} \lambda_t b_t = 0 \tag{3.66}$$

Notice that we take the first order condition with respect to b_{t+1}. This is only valid if the optimal b_{t+1} is in the interior of its feasible set (i.e., it is finite). If we were being careful, we would prove that was the case. Instead, let me just assure you it is.

These conditions imply that:

$$\lambda_t = \theta_t = \beta^t u'(c_t) \tag{3.67}$$

They also imply that:

$$R_{t+1} = r_{t+1} + 1 - \delta \tag{3.68}$$

In other words, the rate of return on the financial asset and physical capital must be equal in order for the consumer's holdings of both to lie in the interior of his feasible set. Equation (3.68) is a *no-arbitrage* condition. We will use this type of condition again and again in the pricing of assets. The essence is to recognize that violations of this condition produce infeasible demand levels. If $r_{t+1} > R_{t+1}$, a consumer could make arbitrarily large sums of money by borrowing at R_{t+1}, and investing at r_{t+1}.

Combining equations, we get:

$$\beta u'(c_{t+1}) R_{t+1} = u'(c_t) \tag{3.69}$$

$$\beta u'(c_{t+1})(r_{t+1} + 1 - \delta) = u'(c_t) \tag{3.70}$$

This is the Euler equation for this economy. Let's compare this to the Euler equation from the planner's problem:

$$\beta u'(c_{t+1}) f'(k_{t+1}) = u'(c_t) \tag{3.71}$$

So the two Euler equations will imply the same law of motion for consumption if:

$$r_t = f'(k_t) - (1 - \delta) \tag{3.72}$$

Let's move on to the firm's problem to see if this is the case.

The firm

The firm is competitive and has a CRS production function. This means all inputs are paid their marginal products, and profits are zero.

$$r_t = \frac{\partial F(k, L)}{\partial k} = f'(k_t) - (1 - \delta) \tag{3.73}$$

$$w_t = \frac{\partial F(k, L)}{\partial L} = y_t - r_t k_t \tag{3.74}$$

$$\pi_t = 0 \tag{3.75}$$

Comparison between equilibrium and planner's problem

So the equilibrium can be characterized by several equations:

$$k_{t+1} = f(k_t) - c_t \tag{3.76}$$

$$\beta u'(c_{t+1}) f'(k_{t+1}) = u'(c_t) \tag{3.77}$$

$$\lim_{t \to \infty} \beta^t u'(c_t) k_t = 0 \tag{3.78}$$

$$k_0 > 0 \quad \text{given} \tag{3.79}$$

These are exactly the equations that describe the planner's problem! So the dynamics will be exactly the same, the steady state will be exactly the same, and so on.

Since we know a unique solution exists to the planner's problem, we know that this economy has a unique equilibrium.

3.3 Fiscal policy in the RA model

Main questions of fiscal policy:

1. What is the impact of an increase in spending (accompanied by an equal increase in taxes)?
2. What is the impact of an increase in the budget deficit (lowering taxes while keeping spending constant)?
3. What is the impact of various tax regimes?

3.3.1 Lump-sum taxation

We add another agent to our model.

Government

Spending: Let g_t be government consumption. To keep it simple we make three assumptions:

1. g_t is exogenous
2. $g_t = g$.
3. Government spending is thrown away (alternatively it could enter into the utility function or production function)

Taxes: The government collects lump-sum taxes T_t. To keep things simple, we impose a balanced budget so that $T_t = g$.

Consumer

The consumer pays taxes, changing his problem to that of maximizing:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3.80}$$

subject to the budget constraint:

$$c_t + k_{t+1} + b_{t+1} + T_t \leq w_t + r_t k_t + R_t b_t + \pi_t \tag{3.81}$$

plus the nonnegativity constraint on capital and the NPG condition. We've assumed 100% depreciation again.

Firm

The firm's problem is the usual.

$$\pi_t = \max_{k_t, L_t} F(k_t, L_t) - r_t k_t - w_t L_t \tag{3.82}$$

Equilibrium

Equilibrium is defined as usual.

Solution

So the Lagrangian is

$$\begin{aligned} L &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &+ \lambda_t[w_t + r_t k_t + R_t b_t + \pi_t - c_t - T_t - b_{t+1} - k_{t+1}] \end{aligned} \tag{3.83}$$

The first order conditions are:

$$\beta_t u'(c_t) - \lambda_t = 0 \tag{3.84}$$
$$r_{t+1}\lambda_{t+1} - \lambda_t = 0 \tag{3.85}$$
$$R_{t+1}\lambda_{t+1} - \lambda_t = 0 \tag{3.86}$$
$$w_t + r_t k_t + R_t b_t - c_t - T_t - b_{t+1} - k_{t+1} = 0 \tag{3.87}$$

and the transversality condition is as before.

The Euler equation is:

$$\begin{aligned} r_{t+1}\beta u'(c_{t+1}) &= u'(c_t) \qquad \text{or} \\ f'(k_{t+1})\beta u'(c_{t+1}) &= u'(c_t) \end{aligned} \tag{3.88}$$

Notice that taxes do not enter into the Euler equation. We find the steady state as before:

$$f'(k_\infty) = 1/\beta \tag{3.89}$$

so steady state capital is unaffected by government spending. However, steady state consumption is:

$$c_\infty = f(k_\infty) - k_\infty - g \tag{3.90}$$

so in the long run, government spending crowds out individual consumption on a one-for-one basis. However, it does not provide any long-run disincentive to accumulating capital.

3.3.2 Distortionary taxation

In the real world, of course, taxes are not lump-sum. Suppose instead, we tax people proportionally to their income.

Government

Again, spending is constant at $g_t = g$.

The government collects taxes equal to some fraction τ_t of each person's income. Again we impose a balanced budget condition:

$$\tau_t = \frac{g}{f(k_t)} \tag{3.91}$$

Notice that no matter how much income the representative agent has, he will pay the same total amount in taxes. However, we're going to assume he doesn't know this and instead *takes the tax rate as given.*

Consumers

The consumer pays taxes, changing his problem to that of maximizing:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3.92}$$

subject to the budget constraint:

$$c_t + k_{t+1} + b_{t+1} \leq (1 - \tau_t)(w_t + r_t k_t + R_t b_t + \pi_t) \tag{3.93}$$

plus the nonnegativity constraint on capital and the NPG condition.

Firm

The firm's problem is the same as before.

Equilibrium

Equilibrium in this economy is a set of prices $\{r_t, w_t\}$, tax rates $\{\tau_t\}$, bond holdings $\{b_t\}$ and resource allocations $\{c_t, k_t\}$ such that:

1. Taking prices and tax rates as given, the consumer selects bond holdings and allocations to solve his maximization problem.
2. Taking prices as given, the firm selects allocations to solve its maximization problem
3. At the given resource allocation levels, the tax rates $\{\tau_t\}$ balance the government budget.
4. All markets clear (i.e., the allocations chosen by the firm and consumers are the same, and the $sb_t = 0$).

Solution

The way we solve the consumer's problem is that we take first order conditions while treating the tax rate as exogenous. The Lagrangian is

$$\begin{aligned} L &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &+ \lambda_t[(1-\tau_t)(w_t + r_t k_t + R_t b_t + \pi_t) - c_t - b_{t+1} - k_{t+1}] \end{aligned} \tag{3.94}$$

The first order conditions are:

$$\beta_t u'(c_t) - \lambda_t = 0 \tag{3.95}$$

$$(1-\tau_{t+1}) r_{t+1} \lambda_{t+1} - \lambda_t = 0 \tag{3.96}$$

$$(1-\tau_{t+1}) R_{t+1} \lambda_{t+1} - \lambda_t = 0 \tag{3.97}$$

$$(1-\tau_t)(w_t + r_t k_t + R_t b_t - c_t) - b_{t+1} - k_{t+1} = 0 \tag{3.98}$$

and the transversality condition is as before.

The Euler equation is:

$$\begin{aligned} (1-\tau_{t+1}) r_{t+1} \beta u'(c_{t+1}) &= u'(c_t) \qquad \text{or} \\ (1-\tau_{t+1}) f'(k_{t+1}) \beta u'(c_{t+1}) &= u'(c_t) \end{aligned} \tag{3.99}$$

Notice that taxes now enter into the Euler equation. Why? Because income taxes reduce the private returns to capital, discouraging investment. This is Economics 101 - if you charge people for doing something (in this case, accumulating capital), they don't do it as much.

The steady state is characterized by:

$$(1-\tau)f'(k_\infty) = \beta^{-1} \tag{3.100}$$

where $\tau = \frac{g}{f(k_\infty)}$. We can substitute back in to get:

$$\left(1 - \frac{g}{f(k_\infty)}\right) f'(k_\infty) = \beta^{-1} \tag{3.101}$$

We can do this any time *after* we take first order conditions. Increased government spending reduces output in the long-run, and reduces consumption by more than in the lump-sum case. Since we know the previous model is Pareto efficient and the allocations differ, this one is inefficent.

3.3.3 Deficit spending

Traditional (Keynesian) view: Deficit spending pumps up the economy by stimulating aggregate demand. However, increased issuance of bonds may crowd out private investment.

Even more traditional view (Ricardian equivalence): A budget deficit has no impact on real economic activity.

Government

Spending: Exogenous sequence g_t.

Taxes: Sequence of lump sum taxes T_t. No balanced budget constraint.

Bonds: Government can issue bonds. One-period budget constraint is:

$$g_t + R_t b_t \leq T_t + b_{t+1} \tag{3.102}$$

Note that from the government's point of view, b_t represents net debt (the opposite of what it means to consumers). While the government need not balance the budget every period, we also impose an NPG condition:

$$\lim_{t\to\infty} q_t b_t = 0 \tag{3.103}$$

where:

$$q_t = \prod_{i=1}^{t} \left(\frac{1}{R_i} \right) \tag{3.104}$$

Consumers

The consumer's problem is the same as before.

Firms

The firm's problem is the same as before.

Equilibrium

The equilibrium description is the same as before.

Solution

We're not going to solve this model directly. Instead we will prove that this model exhibits Ricardian equivalence. How can we do this?

First we replace the sequence of government budget constraints with a single present-value constraint. The one-period constraint is:

$$b_{t+1} = R_t b_t + g_t - T_t \tag{3.105}$$

substituting recursively, we get:

$$b_{t+1} = R_t(g_{t-1} - T_{t-1} + R_{t-1} b_{t-1}) + g_t - T_t \tag{3.106}$$

Multiplying both sides by q_t:

$$R_{t+1} q_{t+1} b_{t+1} = q_{t-1} R_{t-1} b_{t-1} + q_{t-1}(g_{t-1} - T_{t-1}) + q_t(g_t - T_t) \tag{3.107}$$

If we keep substituting, we eventually get:

$$R_{t+1} q_{t+1} b_{t+1} = R_0 b_0 + \sum_{i=1}^{t} q_i (g_i - T_i) \tag{3.108}$$

Let's take the limit of both sides as t approaches infinity. Since $\lim_{t\to\infty} q_t b_t = 0$ and R_{t+1} is bounded for all t.

$$0 = R_0 b_0 + \sum_{i=1}^{\infty} q_t(g_t - T_t) \tag{3.109}$$

or

$$\sum_{i=1}^{\infty} q_t T_t = R_0 b_0 + \sum_{i=1}^{\infty} q_t g_t \tag{3.110}$$

This is the present value budget constraint. It says the present value of future spending (including interest on any period zero debt) and future taxes must be equal.

Now let's do the same for consumers:

$$b_{t+1} + k_{t+1} = w_t + r_t k_t + R_t b_t - c_t - T_t \tag{3.111}$$

Since $r_t = R_t$ in equilibrium, let's temporarily simplify the notation by letting $\hat{b}_t = b_t + k_t$:

$$\hat{b}_{t+1} = (w_t - c_t - T_t) + R_t \hat{b}_t \tag{3.112}$$

Using the same tricks as before:

$$R_{t+1} q_{t+1} \hat{b}_{t+1} = R_0 \hat{b}_0 + \sum_{t=0}^{t} q_t(w_t - c_t - T_t) \tag{3.113}$$

substituting back in, we get:

$$R_{t+1}(q_{t+1} b_{t+1} + q_{t+1} k_{t+1}) = R_0 \hat{b}_0 + r_0 k_0 + \sum_{t=0}^{t} q_t(w_t - c_t - T_t) \tag{3.114}$$

Taking the limit on both sides, and remembering that

$$\lim_{t\to\infty} q_t b_t = 0 \quad \text{by the NPG condition} \tag{3.115}$$

$$\lim_{t\to\infty} q_t k_t = 0 \quad \text{by the TVC condition} \tag{3.116}$$

we get

$$0 = R_0 \hat{b}_0 + r_0 k_0 + \sum_{t=0}^{\infty} q_t(w_t - c_t - T_t) \tag{3.117}$$

or

$$R_0 \hat{b}_0 + r_0 k_0 + \sum_{t=0}^{\infty} q_t w_t = \sum_{t=0}^{\infty} q_t c_t + \sum_{t=0}^{\infty} q_t T_t \tag{3.118}$$

This is the consumer's present value budget constraint. It shows that the present value of all assets (including labor) must equal the present value of future spending. We can substitute to get

$$R_0\hat{b}_0 + r_0k_0 + \sum_{t=0}^{\infty} q_t w_t = \sum_{t=0}^{\infty} q_t c_t + [R_0 b_0 + \sum_{t=0}^{\infty} q_t g_t] \tag{3.119}$$

or

$$r_0k_0 + \sum_{t=0}^{\infty} q_t w_t = \sum_{t=0}^{\infty} q_t c_t + \sum_{t=0}^{\infty} q_t g_t \tag{3.120}$$

Consider a situation in which the budget is balanced. Now suppose the government considers a different sequence of taxes $\{T_t\}$ (which could include deficits or surpluses, but satisfies the basic conditions) and wants to know its impact. Notice that neither the specific sequence of taxes nor the initial debt level affects the consumer's budget constraint. They don't enter into the utility function either. So (for a given set of prices) the consumer's optimal allocation doesn't change when tax policy changes. Since the optimal allocation clears markets at the same prices whatever the tax policy, the equilibrium prices and quantities have not changed. This is Ricardian equivalence.

Where's the economics? It's actually quite simple. In a balanced budget economy, the demand for bonds is zero at the market interest rate R^{bb}. Suppose that the government reduces my tax burden by one dollar today, and issues a one dollar bond that pays interest rate R^d. The supply of bonds has gone up by one. Since my demand for bonds was zero before, one would imagine the interest rate must rise. However, I know the government will tax me R^d tomorrow to pay off this bond. If $R^d = R^{bb}$ I can buy the bond from the government and duplicate my real allocations from the balanced budget economy. My demand for bonds is now one at the market interest rate. Since supply and demand have gone up by one, the equilibrium price is unchanged $R^{bb} = R^d$.

How do I know I would want to duplicate my real allocations from the balanced budget economy? By taxing me one dollar less today and R dollars more tomorrow, the government is giving me a loan at the market interest rate. Since I didn't take such a loan out before, revealed preference indicates I don't want it. If I could, I would choose to make an offsetting loan of one dollar at the market interest rate. Buying the government bond allows me to do just that.

Does Ricardian equivalence hold?

Consensus: The economy does not seem to display Ricardian equivalence.

Why might it break down?

1. Consumers wish to take a loan at the market interest rate but something keeps them from doing so.

   (a) Maybe they want to leave debt to their children. Doubtful (Barro 1974) - If they wish to leave debt to their children, but are constrained from doing so, they would choose to leave zero net bequest. Most parents make positive bequests.

   (b) Maybe they want to borrow (say, to finance education), but face an interest rate premium due to adverse selection problems. Controversial.

2. Taxes are distortionary. True - to see how this would matter, suppose that we have zero taxes in every month but December. In December, we collect taxes based on December's income. What would you do? You wouldn't work in December!

3. People don't really optimize over a long horizon. Likely, but controversial, and a little cheap.