Chapter 8

Search models

In all the models we have considered so far, the market itself is quite simple. Agents observe prices, and these prices are determined by market-clearing.

Search models are a little different. In a search model an individual only has access to some market transactions at a given point in time, and must decide whether to accept the currently available offers, or delay in hopes of receiving a better offer in the future. These models are most commonly used to model unemployment, but also find applications in explaining money demand, generating coordination failures, etc.

8.1 A basic model

Consider a single worker who wishes to maximize discounted lifetime income:

$$\sum_{t=0}^{\infty} \beta^t y_t \tag{8.1}$$

The worker is in one of two states: employed or unemployed, and starts out unemployed. In each period of unemployment, the worker is matched with a single employer who offers wage w_t. The wage offered by the employer is an independent draw from a probability distribution with CDF $F(.)$, where two conditions hold

$$\begin{aligned} F(0) &= 0 \\ F(W) &= 1 \quad \text{for some } W < \infty \end{aligned} \tag{8.2}$$

This will be a partial equilibrium model and we will not explicitly model why there is variation in wage offers. In general equilibrium models, this is usually explained by adding a "matching" feature to the model, i.e., worker-firm pairs are heterogeneous in productivity. In some cases, it can also be explained by a firm taking advantage of the fact that there are economic rents to a match.

Suppose that the worker receives wage offer w_T in period T. The worker can choose to accept or reject the offer. If the worker rejects the offer, he or she is unemployed and receives nothing $y_T = 0$. If the worker accepts a wage offer, he or she continues to work at that wage forever: $y_t = w_T$ for all $t \geq T$.

We can describe the worker's problem by a simple Bellman equation. Let $V(w_0)$ be the value function of an unemployed worker who has received wage offer w_0. If the wage offer is accepted, then

$$V(w_0) = \sum_{t=0}^{\infty} \beta^t w_t = \frac{w_0}{1-\beta} \tag{8.3}$$

If it is rejected, then

$$V(w_0) = \beta E_0(V(w_1)) \tag{8.4}$$

Putting these two things together, the Bellman equation is:

$$\begin{aligned} V(w_t) &= \max\left\{\frac{w_t}{1-\beta}, \beta E(V(w_{t+1}))\right\} \\ &= \max\left\{\frac{w_t}{1-\beta}, \beta \int_0^W E(V(w_{t+1}))dF(w_{t+1})\right\} \end{aligned} \tag{8.5}$$

The $\beta E(V(w_{t+1}))$ term is sometimes called the continuation value.

Now, how do we solve this thing? The first thing to notice is that this type of problem will usually be solved by a reservation wage strategy. That is, there is some r such that we will be indifferent between accepting and rejecting if $w_t = r$, will reject all offers below r and will accept all offers above r. This is something that, if you were writing a paper, must be proved. But for our purposes it is good enough to conjecture that the solution will take this form.

Given this, it will be the case that:

$$V(w_t) = \left\{ \begin{array}{ll} \frac{r}{1-\beta} & \text{if } w_t \leq r \\ \frac{w_t}{1-\beta} & \text{if } w_t \geq r \end{array} \right\} \tag{8.6}$$

Therefore, r must solve:

$$V(r) = \frac{r}{1-\beta} = \beta \left[F(r)\frac{r}{1-\beta} + \int_r^W \frac{w_{t+1}}{1-\beta} dF(w_{t+1}) \right] \tag{8.7}$$

Once we have parameterized the distribution $F(.)$ solving the above equation for r is just an exercise in algebra.

We can also characterize the behavior of employment in this model. Conditional on being unemployed at the beginning of period t, the worker has probability $F(r)$ of being unemployed at the end of the period and probability $1-F(r)$. The probability of being unemployed at the beginning of period t is thus $F(r)^t$.

What does the wage distribution look like?

$$G(w) = \max\left\{ \frac{F(w)-F(r)}{1-F(r)}, 0 \right\} \tag{8.8}$$

8.1.1 An example

For example suppose that the distribution of wage offers is uniform across the $[0, W]$ interval.

$$F(w) = \begin{cases} 0 & \text{if } w < 0 \\ w/W & \text{if } 0 < w < W \\ 1 & \text{if } w > W \end{cases} \tag{8.9}$$

In this case:

$$\begin{aligned} \frac{r}{1-\beta} &= \beta \left[\frac{r}{W}\frac{r}{1-\beta} + \int_r^W \frac{w}{W(1-\beta)} dw \right] \\ &= \frac{\beta}{1-\beta}\left[\frac{r^2}{W} + \frac{W^2 - r^2}{2W} \right] \end{aligned} \tag{8.10}$$

Rearrange and apply the quadratic formula (taking the solution that lies in $[0, 1]$, we get:

$$r = \frac{1-\sqrt{1-\beta^2}}{\beta} W \tag{8.11}$$

So what do we get out of this? First, note that the reservation wage increases proportionally. Second, notice that the reservation wage is increasing in β (you would have to take a derivative to demonstrate this, of course).

8.2 Extensions

The model above is extremely simple, and there are many ways to augment it.

8.2.1 Multiple offers

Suppose that the unemployed worker receives k offers per period, each drawn independently from a distribution with CDF $F(.)$. He or she will only be interested in the largest offer, so let $\tilde{F}(.)$ be the CDF of the largest offer:

$$\tilde{F}(w) = F(w)^k \tag{8.12}$$

In other words, a situation in which the worker receives k offers from distribution $F(.)$ will yield the same behavior as one where he or she receives one offer from distribution $\tilde{F}(.)$.

The same approach can be used to handle an uncertain number of offers. Letting p_k be the probability of receiving k offers:

$$\tilde{F}(w) = \sum_{k=0}^{\infty} p_k F(w)^k \tag{8.13}$$

8.2.2 Costly entry

Suppose that the unemployed worker must pay a one-time entry cost into the search market of e. The worker will choose to do so if:

$$E(V(w_0)) > e \tag{8.14}$$

8.2.3 Job loss

Now suppose that in each period of employment the worker loses his or her job with an exogenous probability α. This is a useful feature because it allows for there to be a long-run unemployment probability different from zero.

First, let's just assume that the reservation wage is r and the distribution of wage offers is given by the CDF $F(.)$. Let U_t be the probability of being unemployed at time t. Conditional on being employed, the probability of being unemployed in the next period is α. Conditional on being unemployed, that probability is $F(r)$. Therefore the unemployment probability follows the law of motion:

$$U_{t+1} = F(r)U_t + \alpha(1 - U_t) \tag{8.15}$$

We can find the long-run unemployment probability by solving the equation:

$$U = F(r)U + \alpha(1 - U) \tag{8.16}$$

for U, giving:

$$U = \frac{\alpha}{1 + \alpha - F(r)} \tag{8.17}$$

Of course, the reservation wage itself will be a function of α.

8.2.4 Unemployment insurance

For example suppose that the distribution of wage offers is uniform across the $[0, W]$ interval.

$$F(w) = \begin{cases} 0 & \text{if } w < 0 \\ w/W & \text{if } 0 < w < W \\ 1 & \text{if } w > W \end{cases} \tag{8.18}$$

In this case:

$$\begin{aligned} \frac{r}{1-\beta} &= c + \beta \left[\frac{r}{W} \frac{r}{1-\beta} + \int_r^W \frac{w}{W(1-\beta)} dw \right] \\ &= c + \frac{\beta}{1-\beta} \left[\frac{r^2}{W} + \frac{W^2 - r^2}{2W} \right] \end{aligned} \tag{8.19}$$

Rearrange and apply the quadratic formula (taking the solution that lies in $[0, 1]$, we get:

$$r = \frac{1 - \sqrt{1 - \beta^2 - 2\beta(1-\beta)\frac{c}{W}}}{\beta} W \tag{8.20}$$

So what do we get out of this? First, note that increased unemployment compensation (relative to the distribution of wages) tends to raise the reservation wage. Second, note that if unemployment compensation and the wage distribution increase proportionally, there will be a proportional increase in the reservation wage. The next question is whether the reservation wage is increasing or decreasing in β. Suppose $c = 0$; then if you take a derivative, you find that r is increasing in β.

Chapter 9

Endogenous growth

The neoclassical/Solow growth model has 2 big problems

- The sole engine of permanently sustainable growth in the model is "technical progress," an exogenous variable. The model has nothing to say about the determinants of technical progress, so in the end it has almost nothing to say about the determinants of growth.
- The model's empirical predictions are at odds with the data. Specifically, it either implies that rich and poor countries will converge and/or that every country will eventually settle down to the same long run growth rate. Not much evidence for either, though there is this idea of "conditional convergence."

Researchers in the "endogenous growth" literature have tried to rectify these two issues by constructing models in which the long run growth rate is determined by incentives and other "fundamental" influences (rather than some unexplained residual).

9.1 The "AK" model

The "AK" model is the simplest possible endogenous growth model. The consumer has the usual utility function with CRRA preferences (we will do this in continuous time, as is usually the case in the growth literature):

$$U = \int_0^\infty e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} \tag{9.1}$$

subject to the budget constraint:

$$\dot{k}_t = r_t k_t + \pi_t - c_t - \delta k_t \tag{9.2}$$

and the initial condition $k_0 > 0$ given.

The firm is competitive and hires capital k_t to produce output Ak_t. Firm profits are given by:

$$\pi_t = \max_{k_t} \{Ak_t - r_t k_t\} \tag{9.3}$$

I won't bother defining equilibrium, I'll just describe the results and the intuition.

The rental rate on capital is obviously:

$$r_t = A \tag{9.4}$$

The consumer's Euler equation is:

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= \frac{r(t) - \delta - \rho}{\sigma} \\ &= \frac{A - \delta - \rho}{\sigma} \end{aligned} \tag{9.5}$$

This gives us the growth rate of consumption, which turns out to also be the growth rate of output and capital.

A few things to note:

- The economy is on the balanced growth path immediately.
- The growth rate is an endogenous function of preference and technology parameters. It is constant, so there is no convergence of rich and poor economies.
- The key difference between this model and the Solow model is that it has constant returns to scale in capital. In the Solow model, capital has decreasing returns, so eventually its contribution to growth disappears. In general, endogenous growth models will have the feature that they exhibit constant returns to scale in *accumulable* or *reproducible* factors of production.

- That said, it has a distinctly unpalatable feature: there is no labor in this model. This almost has to be the case for there to be CRS in capital (necessary to generate endogenous growth) and CRS overall (necessary for firms to behave competitively). Another way of saying this is that capital's share of output here is 100%. In the real world however, 2/3 of factor payments are wages!

9.1.1 A model with physical and human capital

One way that we can make an "AK"-style model without having unreasonable implications about factor shares is to interpret wage payments as payments to human capital (a reproducible factor) rather than as payments to simple labour (a fixed factor). In this case the production function is:

$$Y_t = F(K_t, H_t) \tag{9.6}$$

where K_t is capital and H_t is human capital; and consumers can invest resources in accumulating either type of capital.

Even so, we run into the problem that CRS in reproducible factors implies either that there are no fixed factors (or at least that their factor shares are going to zero), or that there are increasing returns in total. Increasing returns are inconsistent with competitive markets.

9.2 A model with spillovers

The first approach is to have a model with spillovers in production. Suppose that there is a large set of firms, each firm indexed by i. Let $L(i)$, $K(i)$, and $Y(i)$ be labour, capital, and output for firm i, and let L, K, and Y be aggregate labour, capital, and output.

$$\begin{aligned} L &\equiv \sum_i L(i) \\ K &\equiv \sum_i K(i) \\ Y &\equiv \sum_i Y(i) \end{aligned}$$

The production function for an individual firm is given by:

$$Y(i) = BK(i)^{\alpha} L(i)^{1-\alpha} K^{\lambda} \tag{9.7}$$

where $\alpha \in (0, 1)$ and $\lambda \geq 0$. The firm's productivity is increasing in aggregate capital K, which each firm takes as given. The idea here is that there is an agglomeration or scale externality: firms are more productive when activity in the economy is high.

From the point of view of each firm, production is CRS. As a result, each firm has finite factor demands as long as factor prices are equal to marginal productivity (taking K as given). However, the aggregate production function exhibits increasing returns to scale for $\lambda > 0$:

$$\begin{aligned} Y &= \sum_i Y(i) \qquad (9.8) \\ &= \sum_i BK(i)^{\alpha} L(i)^{1-\alpha} K^{\lambda} \\ &= \sum_i BK^{\lambda} \frac{K}{L}^{\alpha} L(i) \\ &= BK^{\alpha+\lambda} L^{1-\alpha} \end{aligned}$$

Note that we made use in the of an earlier result of ours that each firm here chooses the same capital to labour ratio.

If $\alpha + \lambda = 1$, then this model collapses into an "AK" model, where $A = BL^{1-\alpha}$. Notice that in this case the growth rate is increasing in the size of the population (L). Many but not all endogenous growth models have this type of "scale effect."

9.3 A model with endogenous technology

Our next model will be a simplified version of Paul Romer's (1990 JPE) model of endogenous technological progress. It features some modeling tricks used frequently in macroeconomic theory to model product differentiation and imperfect competition.

9.3.1 Consumers

Consumers have a CRRA utility function. There is no capital in this model, so the intertemporal element of choice will be in the bond market. The consumer is endowed with one unit of labour per period, and chooses consumption and bond purchases to maximize:

$$U = \int_0^\infty e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt \tag{9.9}$$

subject to the budget constraint:

$$\dot{b}_t + c_t \leq w_t + r_t b_t + \int_0^{A_0} \pi_t(i)$$

plus the no-Ponzi-game condition and some initial bond holdings b_0. The variable π_t represents the profits of all firms owned by the consumer. The consumer owns all firms in existence at time zero.

There are three sectors in the economy: the final goods sector, the intermediate goods sector, and the research and development sector. The worker supplies labour to firms in the intermediate goods sector and the R&D sector:

$$L_t^A + \int_0^\infty L_t(i) = 1 \tag{9.10}$$

where L_t^A is labour supplied to the R&D sector and $L_t(i)$ is labour supplied to firm i in the intermediate goods sector.

9.3.2 Intermediate goods producers

In each period t there is a continuum of intermediate goods producers indexed by $i \in [0, A_t]$. Firm i is the monopoly producer of good i, and good i is distinct from any other good. Firm i's output is CRS in labour:

$$x_t(i) = L_t(i) \tag{9.11}$$

where $x_t(i)$ is firm i's output of good i, and $L_t(i)$ is the firm's labour input. The firm is a monopolist in its good, and so is *not* a price taker.

Let $p_t(i)$ be the market price of good i, and let $p_t(i)[x_t(i)]$ be the market demand curve for good i. Then firm i's problem is given by

$$\max_{x_t(i)} \{p_t(i)[x_t(i)] * x_t(i) - w_t x_t(i)\} \tag{9.12}$$

9.3.3 Final goods producers

The final goods sector features a single competitive firm that simply combines intermediate goods into the composite consumption good. Its production function is given by:

$$c_t = \left[\int_0^\infty x_t(i)^\alpha di \right]^{1/\alpha} \tag{9.13}$$

Consumption is the numeraire, so the profit maximization problem of the final goods producer is given by:

$$\max_{x_t(i)} \left\{ \left[\int_0^\infty x_t(i)^\alpha di \right]^{1/\alpha} - \int_0^\infty p_t(i) x_t(i) di \right\} \tag{9.14}$$

9.3.4 Research and development

The research and development sector works like this: as of period t, there are firms producing a variety of intermediate goods $i \in [0, A_t]$. Firms in the R&D sector employ labour and produce new varieties (i.e. they expand A_t). Onced a firm in the R&D sector has created a variety it owns the monopoly right to produce that variety. The production function for new varieties is given by

$$\dot{A}_t = \delta L_t^A A_t \tag{9.15}$$

Firms finance R&D by issuing bonds, and then pay the bonds back with the monopoly profits from producing the new goods. There is free entry in the R&D sector, implying zero profits, i.e., the discounted net present value of the profits from producing new goods will be exactly equal to the amount that must be borrowed to finance the research.

9.3.5 Solving the model

This is a more complex model than we have seen so far, but it is a lot like models that you see in the literature these days, and is even a simplified version of Romer's 1990 paper.

Our first step is to figure out what the consumers do. Set up the Hamiltonian:

$$H = e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} + \lambda_t \left[w_t + r_t b_t + \pi_t - c_t \right] \tag{9.16}$$

Our first order conditions are:

$$\frac{\partial H}{\partial c} = e^{-\rho t}c_t^{-\sigma} - \lambda_t = 0$$
$$\frac{\partial H}{\partial b} + \dot{\lambda}_t = \lambda_t r_t + \dot{\lambda}_t = 0$$

Taking the time derivative of λ_t we get:

$$\dot{\lambda}_t = -\rho e^{-\rho t}c_t^{-\sigma} + -\sigma e^{-\rho t}c_t^{-\sigma-1}\dot{c}_t \tag{9.17}$$

Substituting and solving we get:

$$r_t = \rho + \sigma\frac{\dot{c}_t}{c_t} \tag{9.18}$$

Next, we figure out what the final goods producers do. Taking first order conditions of their profit maximization problem with respect to $x(j)$ for some $j \in [0, A_t]$ we get:

$$\frac{1}{\alpha}\left[\int_0^\infty x_t(i)^\alpha di\right]^{\frac{1-\alpha}{\alpha}} * \alpha x_t(j)^{\alpha-1} - p_t(j) = 0 \tag{9.19}$$

Rearranging, we get:

$$x_t(j) = c_t p_t(j)^{\frac{-1}{1-\alpha}} \tag{9.20}$$

or:

$$p_t(j) = c_t^{1-\alpha}x_t(j)^{\alpha-1} \tag{9.21}$$

This equation gives the demand curve for the intermediate good j.

Now we consider the profit maximization problem for the producer of intermediate good i. Substituting in the demand curve above, we get:

$$\pi_t(i) = \max_x\left\{c_t^{1-\alpha}x_t(i)^{\alpha-1} * x_t(i) - w_t x_t(i)\right\} \tag{9.22}$$

Taking first order conditions we get:

$$\alpha c_t^{1-\alpha}x_t(i)^{\alpha-1} - w_t = 0 \tag{9.23}$$

Solving for $x_t(i)$ we get:

$$x_t(i) = c_t\left(\frac{\alpha}{w_t}\right)^{\frac{1}{1-\alpha}} \tag{9.24}$$

Substituting in we find that:

$$p_t(i) = \frac{w_t}{\alpha} \tag{9.25}$$

and

$$\pi_t(i) = c_t w_t^{\frac{-\alpha}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} \left[\frac{1}{\alpha} - 1\right] \tag{9.26}$$

Notice that neither price nor quantity nor profit depends on i. In other words, each intermediate goods firm makes the exact same price/quantity decision and has the same labour demand. Let L_t^P be the labour input to a representative firm in the intermediate goods sector. Then:

$$L_t^A + \int_0^{A_t} L_t^P di = 1 \tag{9.27}$$

Solving we get:

$$L_t^P = \frac{1 - L_t^A}{A_t} \tag{9.28}$$

We can then substitute this into the production function of the final goods sector to generate:

$$\begin{aligned} c_t &= \left[\int_0^{A_t} \left(\frac{1 - L_t^A}{A_t}\right)^{\alpha} di\right]^{1/\alpha} \\ &= (1 - L_t^A) A_t^{\frac{1-\alpha}{\alpha}} \end{aligned} \tag{9.29}$$

Next, we are going to make a conjecture that the amount labour supplied to the R&D sector is constant over time: $L_t^A = L^A$. If true, this will dramatically simplify the dynamics of the model. First we substitute into the equation above to get:

$$c_t = (1 - L^A) A_t^{\frac{1-\alpha}{\alpha}} \tag{9.30}$$

We can use this equation to calculate the growth rate of consumption as a function of the rate of technical progress:

$$\frac{\dot{c}_t}{c_t} = \frac{1-\alpha}{\alpha} \frac{\dot{A}_t}{A_t} \tag{9.31}$$

We can also substitute into the production function for new designs to get:

$$\frac{\dot{A}_t}{A_t} = \delta L^A \tag{9.32}$$

and finally substitute in to get

$$\frac{\dot{c}_t}{c_t} = \frac{1-\alpha}{\alpha}\delta L^A \tag{9.33}$$

A few things to notice here:

- There is a tradeoff between labour devoted to research and labour devoted to production. The more labour devoted to production $(1 - L^A)$ the more consumption there is for a given level of technology A_t. However, the more labour devoted to research (L^A) the faster is growth in A_t and thus consumption.

- Provided that our conjecture is correct, we start out on the balanced growth path immediately, and our model fits into the "AK" framework, with technology the accumulable factor of production.

- There are scale effects in this model, as the growth rate is increasing in the amount of labour (not just the proportion) supplied to the research and development sector.

Now we need to show that L_t^A really is constant, by evaluating the profit maximization problem of a firm in the R&D sector. This part is more difficult than we need to deal with, so I'll just provide a sketch of the argument. Because there is free entry, profits will always be zero. The discounted-to-period-t net present value of profits in the R&D sector is:

$$\int_t^\infty e^{-\int_t^\tau r(s)ds}\left(\int_{A_t}^{A_\tau} \pi_\tau(i)di - w_\tau L_\tau^A\right) d\tau \tag{9.34}$$

Since this quantity is equal to zero for every t, the derivative with respect to t is also equal to zero. We can take the derivative (remembering both the chain rule and how to take derivatives of an integral), do a little algebra, and get:

$$\pi_t \dot{A}_t = r_t w_t L_t^A \tag{9.35}$$

In other words the current profits from an intermediate goods producer must be exactly enough to pay the interest on the debt incurred to create the new product. We have convenient expressions for everything here except the wage. To get the wage, one convenient approach is to use the zero-profit condition of the final goods firm:

$$\begin{aligned} c_t &= \int_0^{A_t} p_t(i)x_t(i)di \\ &= \int_0^{A_t} \frac{w_t}{\alpha}\frac{1-L_t^A}{A}di \\ &= \frac{w_t}{\alpha}(1-L_t^A) \end{aligned} \tag{9.36}$$

Solving for w_t we get:

$$w_t = \frac{\alpha c_t}{1-L_t^A} = \alpha A_t^{\frac{1-\alpha}{\alpha}} \tag{9.37}$$

If we do a little substitution, we get something like:

$$\begin{aligned} L_t^A &= \frac{\delta(1-\alpha)-\alpha\rho}{\delta(1-\alpha)[\sigma+1]} \\ &= \frac{1}{\sigma+1} - \frac{\alpha\rho}{\delta(1-\alpha)[\sigma+1]} \end{aligned} \tag{9.38}$$

which is in fact a constant, verifying our earlier conjecture.