

Lecture 19 - Collisions and conservation of momentum

Text: similar to Fowles and Cassiday, Chap. 7

In first year physics, if not high school, the idea was introduced that some quantities are conserved during a process such as a collision. By *conserved*, we mean that the value of the quantity does not change during a process. Two kinematic quantities that are conserved in a collision are momentum and total energy. We write conservation of momentum for a 2-body collision.

$$1 + 2 \rightarrow 1 + 2 + \dots N$$

as

$$\mathbf{p}_1 + \mathbf{p}_2 = \sum_i \mathbf{p}'_i \quad (1)$$

where the prime indicates the quantity after the collision. Although total energy is conserved, kinetic energy is not necessarily conserved, and we write the change in kinetic energy as Δ .

$$K_1 + K_2 = \sum_i K'_i + \Delta \quad (2)$$

Aside:

Text uses Q which has the opposite sign to Q - value

$$Q\text{-value} = \sum_i m_i c^2 - \sum_f m_f c^2$$

Q - value > 0 for energy release

But

$$\Delta = Q_{\text{text}} < 0 \text{ for energy release}.$$

2-body collisions in 1 dimension

Here the kinematics is particularly simple since all objects travel on the same axis, which we will choose to be the x -axis. Applying conservation of momentum:

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = m_1 \mathbf{v}'_1 + m_2 \mathbf{v}'_2$$

In the cm frame, the signs of \mathbf{v}_1 and \mathbf{v}_2 would be opposite.

The state after the collision has two unknowns, \mathbf{v}'_1 and \mathbf{v}'_2 , which must be determined by two equations. Conservation of momentum provides only one such equation, the other must be the energy-balance equation with a known value of Δ .

Aside

One quantity of some use in collisions is the coefficient of restitution ε :

$$\varepsilon = \frac{\mathbf{v}'_1 - \mathbf{v}'_2}{\mathbf{v}_1 - \mathbf{v}_2}$$

For perfectly elastic collisions in which there is no loss of kinetic energy,

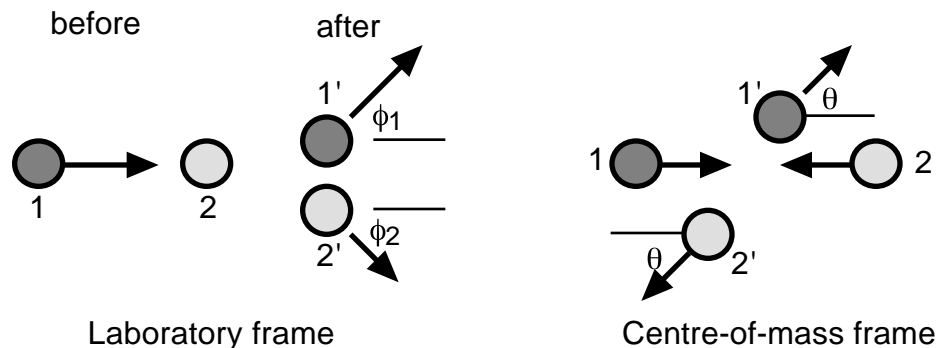
$\Delta = 0$ and $\varepsilon = 1$. In general,

$$\Delta = [\mu(\mathbf{v}_1 - \mathbf{v}_2)^2 / 2](1 - \varepsilon^2)$$

where μ is the reduced mass.

2-body collisions in 2 dimensions

In this situation, the momenta of the objects define a two-dimensional plane. Of interest in many scattering experiments in subatomic physics is the case where object #2 is initially at rest: $\mathbf{p}_2 = 0$. The coordinate system in which object #2 is at rest is referred to as the laboratory system. In contrast, in the centre-of-mass system, all objects are moving, but the total momentum is zero. It is easy to transform from one system to the other.



Conservation of momentum reads

$$\mathbf{p}_1 = \mathbf{p}'_1 + \mathbf{p}'_2 \quad (2 \text{ eqs.})$$

and energy balance reads

$$p_1^2 / 2m_1 = p_1'^2 / 2m_1 + p_2'^2 / 2m_2 + \Delta \quad (1 \text{ eq.})$$

This gives us 3 equations for 4 unknowns (the 4 components of \mathbf{p}'_1 and \mathbf{p}'_2). Hence, we need another piece of information (such as the scattering angle) to solve for the final state. Consider the special case where $m_1 = m_2 = m$. Then the energy balance equation reads

$$p_1^2 = p_1'^2 + p_2'^2 + 2m\Delta \quad (3)$$

But the conservation of momentum equation, when squared, gives

$$\begin{aligned} p_1^2 &= (\mathbf{p}'_1 + \mathbf{p}'_2) \cdot (\mathbf{p}'_1 + \mathbf{p}'_2) \\ &= p_1'^2 + p_2'^2 + 2\mathbf{p}'_1 \cdot \mathbf{p}'_2 \end{aligned}$$

Substituting from Eq. (3) gives

$$2m\Delta = 2\mathbf{p}'_1 \cdot \mathbf{p}'_2$$

or

$$\mathbf{p}'_1 \cdot \mathbf{p}'_2 = m\Delta$$

Now, if $\Delta = 0$ (as it nearly is for billiard balls) then

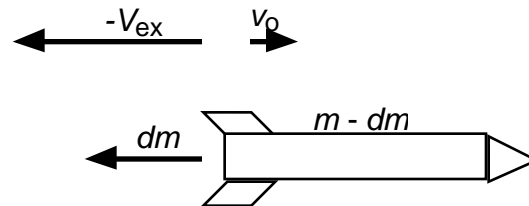
$$\mathbf{p}'_1 \cdot \mathbf{p}'_2 = 0$$

or

$$\mathbf{p}'_1 \text{ is perpendicular to } \mathbf{p}'_2.$$

Rocket equation

The motion of a rocket is a conservation of momentum problem with variable mass. The initial mass of the rocket decreases with time, as exhaust is expelled out the rocket's engines.



We will only consider motion in one dimension. Suppose that we define the velocity of the rocket exhaust relative to the rocket to be a constant V_{ex} . Then, when a mass dm is expelled, the momentum of the rocket changes from

$$mv_0$$

to

$$(m - dm)(v_0 + dv)$$

The momentum of the exhaust is $+(v_0 - V_{ex})dm$ where $V_{ex} > 0$ corresponds to motion away from v_0 .

By conservation of momentum, then

$$mv_0 = (m - dm)(v_0 + dv) + (v_0 - V_{ex})dm$$

Keeping terms to leading order in dm etc. leads to

$$mv_0 = mv_0 - v_0 dm + mdv + v_0 dm - V_{ex} dm$$

or

$$mdv = V_{ex} dm.$$

Some care must be taken with the integration limits when we integrate this expression, since our dm is equivalent to a decrease in mass. Thus

$$\int_{\text{initial}v}^{\text{final}v} dv = V_{ex} \int_{\text{initial}m}^{\text{final}m} m^{-1} dm$$

$$\text{Thus, } v - v_0 = V_{ex} \ln(m_0 / m)$$

What's interesting about this equation is that the change in speed depends logarithmically on the rocket mass. Put another way,

$$m = m_0 \exp [-(v - v_0) / V_{ex}]$$

and the mass decreases exponentially with the velocity of the rocket.