Lecture 2 - Newtonian Mechanics (Fowles and Cassiday, Chap. 2)

Students who have completed first-year physics are already familiar with Newton's law's of motion, which we restate here as

1st Law Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces acting upon it.
2nd Law The acceleration **a** experienced by a body when subject to an unbalanced force **F** is directly proportional to **F** and inversely proportional to the mass *m* of the body.
3rd Law For every action there is an equal and opposite reaction

We've done countless simple examples of these laws previously, concentrating on
i) constant forces
ii) central forces (*i.e.*, forces depending only on radial separation).
Here, we will spend our time on velocity-dependent forces, typically found in viscous media, after briefly reviewing some simple systems.

Constant Force

Constant force implies constant acceleration from **a** = **F**/*m*, leading to simple kinematics:

$$\mathbf{v} = \int \mathbf{a}\,dt \;\text{-->}\; \mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t$$
$$\mathbf{x} = \int \mathbf{v}\,dt \;\text{-->}\; \mathbf{x} = \mathbf{v}_i t + \mathbf{a}t^2/2$$

Position-dependent forces

If the force experienced by an object depends only on its position, then **F** = *m***a** is a differential equation for the position *x*: Working in one dimension:

$$F(x) = m\frac{d^2x}{dt^2} = m\frac{dv}{dt}$$

One can integrate this expression to obtain the conventional expression for work

$$\text{Work} = \int F dx = m\int \frac{dv}{dt}dx$$

$$= m\int \frac{dv}{dx}\frac{dx}{dt}dx = m\int \frac{dv}{dx}v\,dx = m\int v\,dv$$

$$= mv_f^2/2 - mv_i^2/2$$

Or

$$\text{Work} = \Delta K \qquad \text{(we will use } K \text{ for kinetic energy)}$$

Note that the work is independent of the path taken between the initial and final position in this example (because the force is not dissipative). Now, the work done by the system decreases its potential energy *V*. Using the fact that the work done on the system is the negative of the work done by the system, then

$\Delta K = -\Delta V$ or $\Delta(K + V) = 0$ (no dissipation)

We define the energy E as the sum of K and V, via

$$E = K + V = \tfrac{1}{2}mv^2 + V \qquad (2.1)$$

In first year, we solve the motion of an object, x as a function of t, by looking at the forces applied to it. In Eq. (2.1) we have an alternate approach, which gives us t as a function of x by looking at the potential energy. Formally, Eq. (2.1) can be solved for the speed v:

$$v = \pm\left(\frac{2}{m}\left[E - V(x)\right]\right)^{1/2} \qquad (2.2)$$

But $v = dx / dt$, so Eq. (2.2) can be rewritten

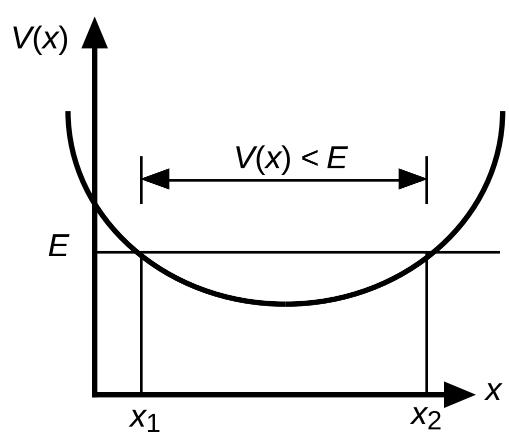
$$\frac{dx}{dt} = \pm\left(\frac{2}{m}\left[E - V(x)\right]\right)^{1/2}$$

$$\pm\int \frac{dx}{\left(\frac{2}{m}\left[E - V(x)\right]\right)^{1/2}} = t - t_o \qquad (2.3)$$

In order for this equation to have a real solution, the square root must be real, or

$V(x) < E$ (for allowed values of x)

Thus, x must be restricted to lie within the classical turning points x_1 and x_2 below:



We now perform two examples to familiarize ourselves with working with potentials.

Example of integrating (2.3):

Find the period T of simple harmonic motion of a spring by integrating (2.3) between $x = -A$ and $x = +A$ for $V(x) = kx^2/2$. The energy E is fixed by the amplitude to be $E = kA^2/2$. This should give $T/2$; the choice of sign for Eq. (2.3) is positive, since $v > 0$ in this domain of x.

$$\frac{T}{2} = \int \frac{dx}{\left(\frac{2}{m}\left[\frac{kA^2}{2} - \frac{kx^2}{2}\right]\right)^{1/2}} = \left(\frac{m}{k}\right)^{1/2} \int_{-A}^{+A} \frac{dx}{\left(A^2 - x^2\right)^{1/2}}$$

Changing variables to $z = x/A$:

--> $$\frac{T}{2} = \left(\frac{m}{k}\right)^{1/2} \int_{-1}^{+1} \frac{dz}{\left(1 - z^2\right)^{1/2}}$$

This can be integrated by trig substitution for $z = \sin\theta$ to give

$$\int_{-1}^{+1} \frac{dz}{\left(1 - z^2\right)^{1/2}} = \int_{-1}^{+1} \frac{d\sin\theta}{\left(1 - \sin^2\theta\right)^{1/2}} = \int_{-\pi/2}^{\pi/2} \frac{\cos\theta \, d\theta}{\cos\theta} = \int_{-\pi/2}^{\pi/2} d\theta = \pi$$

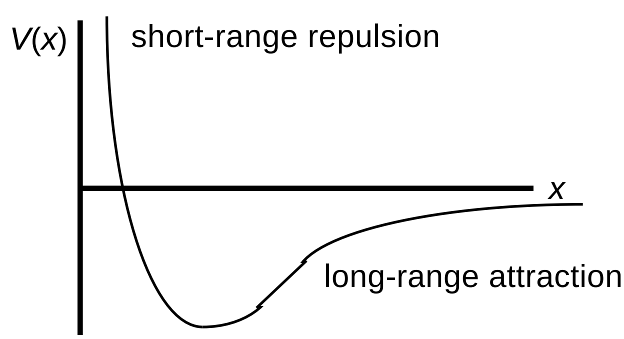
Thus,

$$\frac{T}{2} = \pi\left(\frac{m}{k}\right)^{1/2} \quad \text{or} \quad T = 2\pi\left(\frac{m}{k}\right)^{1/2} \tag{2.4}$$

This is the same expression as we find by setting up the force equation and solving the resulting differential equation for $x(t)$, namely $d^2x/dt^2 = -(k/m)\, x(t)$.

Potential energies for atoms and molecules

There are a number of functional forms that can be used to represent the potential energy between atoms or molecules. They tend to have the generic form



Lennard-Jones	$V(r) = 4V_o\,[(r_o/r)^{12} - (r_o/r)^6]$
(Van der Waal's force between molecules)	$V \to \infty$ as $r \to 0$
Morse (between atoms)	$V(x) = V_o\left\{1 - e^{\frac{x_o - x}{\delta}}\right\}^2 - V_o$
	$V(x) \to V_o e^{\frac{x_o}{\delta}}\left\{e^{\frac{x_o}{\delta}} - 2\right\}$ as $x \to 0$

Both potentials go to zero as $x \to \infty$.

Let's examine the Morse potential in some detail, since most students have done the Lennard-Jones in other courses. The force generated by the Morse potential has a minimum at $x = x_o$, which can be seen by finding the value of x where $dV/dx = 0$. Let's verify this, and find a useful approximation at the same time, by expanding the exponential using $e^z \sim 1+z$ for small z:

$$V(x \approx x_o) \approx V_o\left\{1 - \left(1 + \frac{x - x_o}{\delta}\right)\right\}^2 - V_o = V_o\left\{\frac{x - x_o}{\delta}\right\}^2 - V_o \qquad (2.5)$$

This expression has a minimum at $x = x_o$, where $V(x_o)$ has a value of

$V(x_o) = V_o \bullet (0)^2 - V_o = -V_o$

Eq. (2.5) looks like Hooke's Law $F = -k(x-x_o)$ with a spring constant $k_{sp} = 2V_o/\delta^2$. This is one of many examples where the potential varies quadratically near its equilibrium position, even though it is not quadratic far from equilibrium.

The following are some typical values of the binding energy and spring constant for a few diatomic molecules (data compiled ~ 1970):

	V_o				$k_{sp} = 2V_o/\delta^2$
	(eV)	(J)	x_o(nm)	δ(nm)	(J/m^2)
H_2	4.749	7.60×10^{-19}	0.074	0.0514	5.76×10^{2}
N_2	9.903	1.58×10^{-19}	0.1098	0.0117	2.29×10^{3}
O_2	5.214	8.34×10^{-19}	0.1207	0.0376	1.18×10^{3}
I_2	1.557	2.49×10^{-19}	0.2667	0.054	1.72×10^{2}