Lecture 8 - Forces in three dimensions

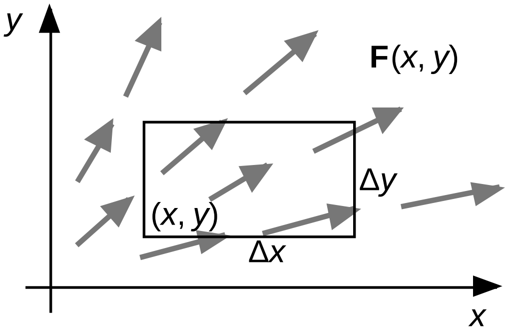
Text: Fowles and Cassiday, Chap. 4

We now revisit some aspects of force in three dimensions. Forces are classified as conservative or nonconservative according to whether their integrals are path-dependent. For example, the force of friction is non-conservative, meaning that the total energy $E = K + V$ is not conserved. Where the path-dependence of a non-conservative force shows up is in the evaluation of a quantity like the work:

Conservative: W depends only on end-points of $\int \mathbf{F} \bullet d\mathbf{r}$
Nonconservative: W depends on integration path

Because the work associated with non-conservative forces is path-dependent, and because $W = \Delta V$, then a potential energy cannot be defined for a non-conservative force: V would depend upon the history of a system, not just its configuration.

So how can we tell mathematically whether a force is conservative without having to evaluate $\int \mathbf{F} \bullet d\mathbf{r}$ for various paths? Consider a rectangular path in the xy-plane. Let the force be a function of x and y on this plane, as in



Around this closed integration loop, the work done on the system is

$$W = \oint \mathbf{F} \bullet \mathrm{d}\mathbf{r}$$

Since the work is equal to the change in potential energy of a conservative force, then the work should vanish if the force is conservative. That is, if the potential energy is a well-defined, path-independent quantity, then a closed line integral around the force field should vanish.

We perform the integral as in the diagram

$$\oint \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \int_{x}^{x+\Delta x} F_x(y)\mathrm{d}x + \int_{y}^{y+\Delta y} F_y(x+\Delta x)\mathrm{d}y + \int_{x+\Delta x}^{x} F_x(y+\Delta y)\mathrm{d}x + \int_{y+\Delta y}^{y} F_y(x)\mathrm{d}y$$

or

$$\oint \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \int_{x}^{x+\Delta x} [F_x(y) - F_x(y+\Delta y)]\mathrm{d}x + \int_{y}^{y+\Delta y} [F_y(x+\Delta x) - F_y(x)]\mathrm{d}y \qquad (8.1)$$

Assume that Δx and Δy are small enough that one can perform a Taylor series expansion about $\mathbf{F}(x, y)$. Note that both F_x and F_y both vary with x and y.

$$F_x(y + \Delta y) = F_x(y) + \Delta y[\partial F_x(y) / \partial y] \qquad (8.2)$$
$$F_y(x + \Delta x) = F_y(x) + \Delta x[\partial F_y(x) / \partial x] \qquad (8.3)$$

Substituting Eqs (8.2) and (8.3) into (8.1) gives

$$\oint \mathbf{F} \bullet \mathrm{d}\mathbf{r} = -\int_{x}^{x+\Delta x} \Delta y[\partial F_x(y) / \partial y]\mathrm{d}x + \int_{y}^{y+\Delta y} \Delta x[\partial F_y(x) / \partial x]\mathrm{d}y$$

By the assumptions of the Taylor series, the terms inside the integrals are constant, leaving us to evaluate terms like

$$\int_{x}^{x+\Delta x} \mathrm{d}x = \Delta x$$

Hence

$$\oint \mathbf{F} \bullet \mathrm{d}\mathbf{r} = \Delta x \Delta y \left[\partial F_y(x) / \partial x - \partial F_x(y) / \partial y\right]$$

This integral is proportional to the enclosed area $\Delta x \bullet \Delta y$. Thus, the condition that the integral vanishes is then

$$\partial F_y(x) / \partial x - \partial F_x(y) / \partial y = 0$$

This result can be generalized to higher dimensions.

First, note that

- $\Delta x \bullet \Delta y \dashrightarrow \int da$ over the area a enclosed by the loop
- $\partial F_y / \partial x - \partial F_x / \partial y$ transforms like a component of a vector, and can be made to look like a vector using $[0, 0, \partial F_y / \partial x - \partial F_x / \partial y]$**.**

Writing this out for a general orientation of the area element becomes

$$[\partial F_z / \partial y - \partial F_y / \partial z,\ \partial F_x / \partial z - \partial F_z / \partial x,\ \partial F_y / \partial x - \partial F_x / \partial y]$$
$$= \text{curl } \mathbf{F} = \ \mathbf{xF}$$

where the del operator is

$$= (\partial / \partial x, \partial / \partial y, \partial / \partial z) = \mathbf{i}(\partial / \partial x) + \mathbf{j}(\partial / \partial y) + \mathbf{k}(\partial / \partial z)$$

and where **i**, **j**, **k** are unit vectors pointing along the Cartesian axes.

curl **F** = $\ \mathbf{xF}$ is a vector (from the cross product)
grad $V = \ V$ is a vector (V is a scalar function)

Hence, $\oint \mathbf{F} \bullet d\mathbf{r} = \int (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\, da$, a result known as Stoke's theorem (**n** is a unit vector normal to a surface bounded by the path of the line integral: $d\mathbf{r} \to \mathbf{n}da$)

OK, so curl **F** vanishes for a conservative force. What does this tell us about the potential energy V? Back in first year, we showed that a non-dissipative force **F** is related to a potential V through

$$F_i = -\partial V / \partial x_i$$

Does this satisfy curl **F** = 0? Just take one component:

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = -\frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y \partial x} = 0$$

If $\mathbf{F} = - \ V$, then it is easy to see that **F** is a conservative force because

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = -\int_A^B \ V \cdot d\mathbf{r} = -\int_A^B dV = -\Delta V$$

Of course the work is also equal to the change in kinetic energy ΔK, and so we have

$\Delta K = -\Delta V$ or $\Delta(K + V) = 0$ or $\Delta E = 0$

Lastly, if the force is dissipative, one cannot write $F = - \ V$. Then, the convention is to break up **F** into two pieces,

$$\mathbf{F} = \mathbf{F}_{\text{cons}} + \mathbf{F}_{\text{non-cons}}$$

so that

$$\int \mathbf{F} \bullet d\mathbf{r} = \int \mathbf{F}_{\text{cons}} \bullet d\mathbf{r} + \int \mathbf{F}_{\text{non-cons}} \bullet d\mathbf{r}$$

for which

$$\int \mathbf{F}_{\text{non-cons}} \bullet d\mathbf{r} = \Delta E \neq 0.$$

Example Is $\mathbf{F} = \mathbf{i}(ax + by^2) + \mathbf{j}cxy$ conservative?

1. Since there is no z - dependence to **F**, then all $\partial F_i / \partial z = 0$.
2. Since $F_z = 0$, then all $\partial F_z / \partial x_i = 0$.
3. Thus, the only possible non-zero component of curl **F** is the z - component:

$\partial F_y / \partial x - \partial F_x / \partial y$
$= \partial(cxy) / \partial x - \partial(ax + by^2) / \partial y$
$= (c - 2b)y$

Hence, curl **F** = 0 if $c = 2b$. The value of a is unimportant. Finding the curl is a much easier procedure than integrating $\mathbf{F} \bullet d\mathbf{r}$ over paths to find a consistent expression for $V(x, y)$.