Appendix A - Eigenvalues and eigenfunctions

What's important:

- eigenvalues and eigenfunctions
- Hermitian operators

Text: Gasiorowicz, Chap. ??

<u>Some Eigenvalue theorems</u>

In our discussion of the hydrogen atom, we made several statements about orthogonality of eigenfunctions and whether there could be simultaneous eigenvalues in specific cases. We now prove these results in general.

1. Eigenfunctions belonging to different eigenvalues of a Hermitian operator are orthogonal. Proof:

$F \equiv$ hermitian operator with eigenfunctions ψ_2 and ψ_j so that

$F\,\psi_i = f_i\psi_i$; $F\psi_j = f_j\,\psi_j$ where $f \equiv$ eigenvalues.

Then

$\langle\psi_j|F|\psi_i\rangle = f_i\langle\psi_j|\psi_i\rangle$

But

$\langle\psi_j|F|\psi_i\rangle = \langle\psi_i|F|\psi_j\rangle^* = f_j^*\langle\psi_i|\psi_j\rangle^* = f_j^*\langle\psi_j|\psi_i\rangle$

Hence,

$f_i\langle\psi_j|\psi_i\rangle = f_j^*\langle\psi_j|\psi_i\rangle$

Since F is hermitian, then f's are real.

$(f_i - f_j)\langle\psi_j|\psi_i\rangle = 0$

Now, the f's are, by hypothesis, different, then

$\langle\psi_j|\psi_i\rangle = 0$

Example

In discussing the spherical harmonics $Y_{\ell m}$ we said that they were eigenfunctions of two operators

$\mathbf{L}^2$ and L_z

Since each of the $Y_{\ell m}$'s has a distinct eigenvalue for each of these operators, then the $Y_{\ell m}$'s must be orthogonal.

2. If two operators F and G commute, then there exists a set of functions which are simultaneous eigenfunctions of both operators. Proof:

We begin with a set of functions ψ_i which are eigenfunctions of F:

$F\psi_i = f_i\psi_i$

(we will assume that all of the f_i are distinct, although this is not required for the proof)
Now

$$GF\psi_i = G(f_i\psi_i) = f_i(G\psi_i).$$

But

$$GF\psi_i = F(G\psi_i)$$

by commutation. Hence, $G\psi_i$ must be a simple multiple of ψ_i in order that

$$F(G\psi_i) = f_i(G\psi_i)$$

$$G\psi_i = g_i\psi_i$$

and ψ_i is a simultaneous eigenfunction of both F and G. (If there is degeneracy, one may have to take linear combinations.)

Examples: Often, the inverse theorem is used. *E.g.*, since L_x, L_y don't commute with L_z, then the $Y_{\ell m}$'s are not simultaneous eigenfunctions of L_x, L_y and L_z.

3. Given a pair of commuting Hermitian operators F and G, and a set of functions ψ_i such that

$$F\psi_i = f_i\psi_i$$

then

$$\langle\psi_i|G|\psi_j\rangle = 0 \text{ unless } f_i = f_j$$

Proof:

$$\langle\psi_i|GF|\psi_j\rangle = f_j\langle\psi_i|G|\psi_j\rangle$$

But

$$\begin{aligned}\langle\psi_i|GF|\psi_j\rangle &= \langle\psi_j|FG|\psi_i\rangle^* \\ &= \langle\psi_j|GF|\psi_i\rangle^* \\ &= f_i\langle\psi_j|G|\psi_i\rangle^* \qquad (f_i \text{ real}) \\ &= f_i\langle\psi_i|G|\psi_j\rangle\end{aligned}$$

$$f_i\langle\psi_i|G|\psi_j\rangle = f_j\langle\psi_i|G|\psi_j\rangle$$

or

$$(f_i-f_j)\langle\psi_i|G|\psi_j\rangle = 0$$

So, either

$f_i-f_j = 0$ and $\langle\psi_i|G|\psi_j\rangle$ is undetermined

or

$f_i-f_j \neq 0$ and $\langle\psi_i|G|\psi_j\rangle = 0$