Lecture 2 - Probabilistic motion

What's important:
- classical *vs*. quantum motion
- probability densities

Text: Gasiorowicz, Chap. 2

Sometimes, when studying difficult material, it is tempting to "cut to the chase": skip some conceptual foundations in favour of working out examples in order to build an intuitive understanding of how a particular formalism or approach works in practice. To some extent, that is part of the strategy of our introductory course on relativity and quantum concepts, or even the first lecture of this course, where we solved the Bohr model for the hydrogen atom, which gives numerically correct results but rests on an analogy with standing waves. In this lecture, we start to examine the foundations more closely, in particular how to incorporate the uncertainty principle in a description of particle motion.

Classical motion

In classical mechanics, we are used to saying that a particle has position x at time t, and travels with a velocity v. In a force-free environment, the momentum is constant from

$$F = dp/dt$$

and, as a consequence, so is the velocity from

$$p = mv.$$

Thus, when $F = 0$, we have

$$x = vt. \qquad (x_o = 0)$$

Now, let's ask the question: what is the probability of finding this moving particle at a position x at time t? Classically, we would answer that the probability is unity at $x = vt$ zero everywhere else. Mathematically, we must work with probability densities $P(x)$, defined such that

$$P(x)\,dx = [\textit{probability of finding particle between } x \textit{ and } x + dx]. \qquad (1)$$

$P(x)$ has the units of inverse length, it is a probability per unit length. Graphically, what we require is

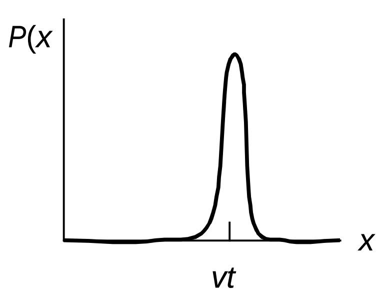


Fig. 1

The delta function $\delta(x)$ provides the representation of this probability density; it is the continuous version of the discrete Kronicker delta δ_{ij},

$$\delta_{ii} = 1 \qquad \delta_{ij} = 0 \quad i \neq j \tag{2}$$

The delta function is

$$\int \delta(x)\, dx = 1 \qquad \delta(x) = 0 \quad x \neq 0 \tag{3}$$

such that

$$\int \delta(x-x_o)\, f(x)\, dx = f(x_o) \tag{4}$$

Although δ_{ii} is unitless, the delta function $\delta(x)$ has units equal to the reciprocal of its argument (here, x). The function is sharply peaked at $x = 0$ to satisfy Eq. (3).

Classically, the momentum is described by a delta function, as it too has a specific value, say p_o:

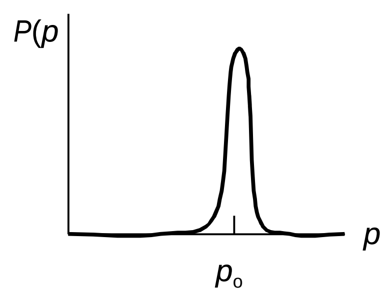


Fig. 2

Thus, we have:

$$P(x) = \delta(x-vt) \tag{5}$$
$$P(p) = \delta(p-p_o)$$

The minus signs assure that the delta function selects the correct values of x and p.

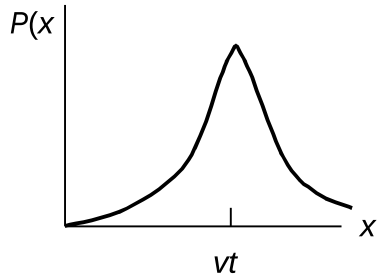
Quantum motion

As described in introductory courses on quantum mechanics, Heisenberg proposed that there is a fundamental limit to how accurately one can simultaneously measure the position and momentum of an object. His uncertainty principle is written as

$$\Delta x \bullet \Delta p \sim \hbar\,, \tag{6}$$

where Δx and Δp are the "uncertainties" in the position and momentum, respectively. It's time to finally confront what Δx means mathematically.

Let's return to Fig. 1. According to Eq. (6), the probability density $P(x)$ must be non-vanishing at positions other than simply $x = vt$: it must be spread out like



How much $P(x)$ is spread depends upon the uncertainty in the momentum: the smaller Δp, the wider the distribution in $P(x)$. Once we define it mathematically, the uncertainty Δx can be extracted from $P(x)$ as follows

$$[\textit{mean value of } x] = x_m = \langle x \rangle = \int P(x)\, x\, dx$$

$$\begin{aligned}\Delta x^2 = \langle (x - x_m)^2 \rangle &= \langle x^2 - 2xx_m + x_m^2 \rangle \\ &= \langle x^2 \rangle - 2x_m \langle x \rangle + \langle x_m^2 \rangle \\ &= \langle x^2 \rangle - 2x_m^2 + x_m^2 \\ &= \langle x^2 \rangle - x_m^2.\end{aligned}$$

What the mathematical formation of quantum mechanics must do is provide us with a means of determining the probability distributions $P(x)$ and $P(p)$. There are a variety of ways to do this: one way works directly with $P(x)$ and measurement operators (covered in PHYS 810 - see Ballentine's book on quantum mechanics; see also Appendix B in Gasiorowicz). Another way is the Schrödinger approach, which determines a complex wave function $\psi(x)$ from which the probability density can be constructed

$$P(x) = \psi^*(x)\psi(x)$$

where ψ^* is the complex conjugate of ψ.