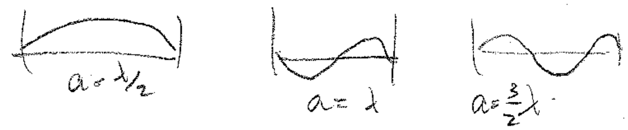


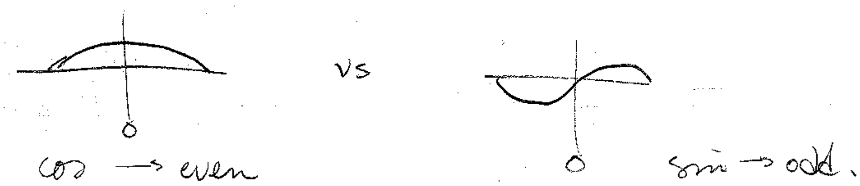
$$a = n\frac{\lambda}{2} \qquad n = 1, 2, 3 \ldots$$

$$\lambda = \frac{2\pi\hbar}{p}$$

$$a = n\frac{\left(\frac{2\pi\hbar}{p}\right)}{2} = \frac{n\pi\hbar}{p}$$

$$\text{or } p_n = \frac{n\pi\hbar}{a} \qquad \Rightarrow \quad \varepsilon_n = \frac{p^2}{2m} = n^2\left(\frac{\pi^2\hbar^2}{2ma^2}\right)$$

The wavefunctions are either odd or even under $x \leftrightarrow -x$.



This symmetry operation has a number of useful properties and is denoted by a symbol $\hat{P}$ (or $\hat{\Pi}$).

$$\hat{P}\,\hat{r} = -\hat{r}\,\hat{P} \qquad \text{where } \hat{r} \text{ is the position operator}$$

$$\Rightarrow \quad [\hat{P}, \hat{r}]_+ = 0 \qquad \text{// anticommutator.}$$

The effect of $\hat{P}$ on $|r\rangle$ is then

$$\underset{①}{\hat{P}}\,\hat{r}|r\rangle = \underset{②}{(r)}\hat{P}|r\rangle$$

$$\| \quad ③ \qquad \text{eigenvalue for } \hat{r}|r\rangle = r|r\rangle$$

$$\Rightarrow -\hat{r}\hat{P}|r\rangle$$

Since ② = ③ $\Rightarrow$ $\hat{r}\{\hat{P}|r\rangle\} = (-r)\{\hat{P}|r\rangle\}$.

i.e. $\hat{P}|r\rangle = |-r\rangle$.

We can find the effects of $\hat{P}$ on ψ by the usual means. Say we have a state "even" under $\hat{P}$:

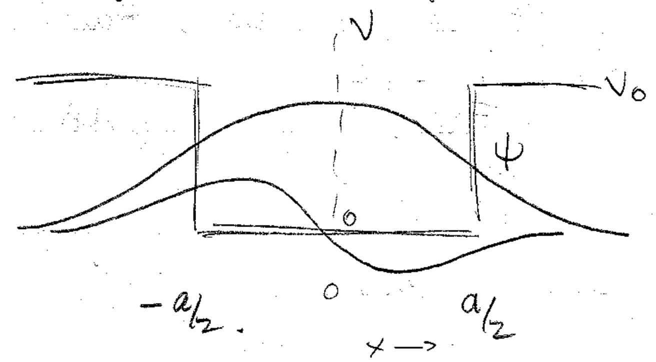
$$\hat{P}\,|\beta_{even}\rangle = +\,|\beta_{even}\rangle .$$

$$\Rightarrow \quad \underbrace{\langle r|\hat{P}|\beta_{even}\rangle}_{\{\langle r|\hat{P}\}|\beta_{even}\rangle \;\to\; \langle -r|\beta_{even}\rangle \;\to\; \psi_{even}(-r)} = +\,\underbrace{\langle r|\beta_{even}\rangle}_{\psi_{even}(r)}$$

implies that $\psi_{even}(r) = \psi_{even}(-r)$.

Finite Symmetric Well

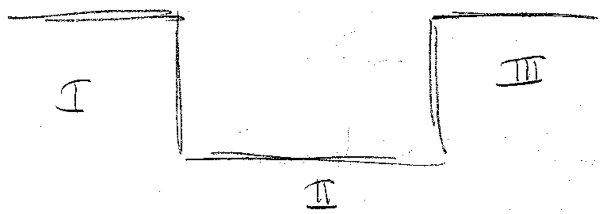
For a finite well, the condition that $\psi = 0$ at the boundaries of the well no longer holds.



We wish to consider the case in which the energy E is less than V_0. The 1-D S.E. reads

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V\psi = E\psi$$

We divide up "space" into 3 regions:



Region II : $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = \epsilon\psi \Rightarrow \frac{d^2}{dx^2}\psi = -\frac{2m\epsilon}{\hbar^2}\psi = -k^2\psi$

$k^2 = \frac{2m\epsilon}{\hbar^2} > 0$

Region I : $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V_0\psi = \epsilon\psi \Rightarrow \frac{d^2}{dx^2}\psi = -\frac{2m}{\hbar^2}(\epsilon - V_0)\psi$

$= \left(\frac{2m}{\hbar^2}(V_0 - \epsilon)\right)\psi$

$= \kappa^2\psi$. $\quad > 0$

Region III : same as region I.

The solutions to S.E. can then be written

$\psi_I = Ce^{\kappa x} + Fe^{-\kappa x}$ (note, $x < 0$ $\therefore$ F term diverges, not C).

$\psi_{II} = A\sin kx + B\cos kx$

$\psi_{III} = De^{-\kappa x} + Ge^{\kappa x}$

Since ψ cannot diverge as $x \to \infty$, then $F = G = 0$. The solutions break into 2 parts, depending on the even or odd nature of ψ_{II}.

$$\psi_{II} \text{ even} \Rightarrow C = D \quad \begin{cases} \psi_{I} = De^{\kappa x} \\ \psi_{II} = B\cos kx \\ \psi_{III} = De^{-\kappa x} \end{cases} \quad \text{even parity}$$

$$\psi_{II} \text{ odd} \Rightarrow C = -D. \quad \begin{cases} \psi_{I} = Ce^{\kappa x} \\ \psi_{II} = A\sin kx \\ \psi_{III} = -Ce^{-\kappa x} \end{cases} \quad \text{odd parity.}$$

To get an equation for the energy, we use the continuity condition on ψ and ψ' at $x = \frac{a}{2}$.

For ψ' : $\frac{d}{dx}\psi_{II}\Big|_{x=\frac{a}{2}} = \frac{d}{dx}\psi_{III}\Big|_{x=\frac{a}{2}}$ ①

on ψ $\quad \psi_{II}(x=\frac{a}{2}) = \psi_{III}(x=\frac{a}{2})$ ②

Dividing ① by ② and substituting

even: $$\frac{-Bk\sin k\frac{a}{2}}{B\cos k\frac{a}{2}} = \frac{-\kappa De^{-\kappa\frac{a}{2}}}{De^{-\kappa\frac{a}{2}}} \Rightarrow k\tan k\frac{a}{2} = \kappa$$

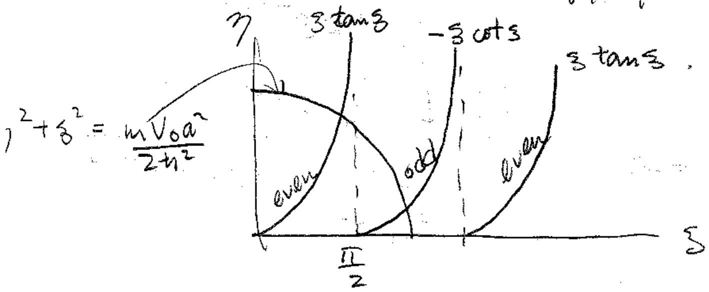
odd: $$\frac{+Bk\cos k\frac{a}{2}}{B\sin k\frac{a}{2}} = \frac{(-C)(-\kappa)e^{-\kappa\frac{a}{2}}}{(-C)e^{-\kappa\frac{a}{2}}} \Rightarrow k\cot k\frac{a}{2} = -\kappa.$$

Now, if we define

$$\eta = \kappa\frac{a}{2} \quad \xi = k\frac{a}{2} \Rightarrow \begin{matrix} \xi\tan\xi = \eta & \text{even} \\ \xi\cot\xi = -\eta & \text{odd.} \end{matrix}$$

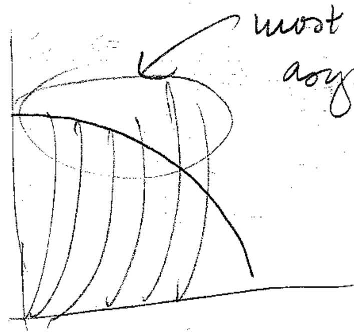
Further $\eta^2 + \xi^2 = \frac{a^2}{4}\left(\frac{2m}{\hbar^2}\right)\left(V_0 - E + E\right) = \frac{mV_0a^2}{2\hbar^2}$

The solutions to the energy equation are at



The number of bound states depends on the product V_0a^2 - However, there must be at least one even bound state in 1-D. This is not true in a 3-D spherical well.

Lastly, the correspondence with the infinite square well can be obtained from the $V_0 \rightarrow \infty$ limit (a fixed).



most of the solutions lie near the asymptotes $n\frac{\pi}{2}$ (for ξ).

$$\Rightarrow \quad \xi = n\frac{\pi}{2}$$

$$k\frac{a}{2} = n\frac{\pi}{2}$$

$$\Rightarrow k^2 = \left(n\frac{\pi}{a}\right)^2$$

$$\frac{2m}{\hbar^2}E = \frac{n^2\pi^2}{a^2}$$

$$E = n^2\frac{\pi^2\hbar^2}{2ma^2}.$$