Lecture 1 - Random walks - I

What's Important:
- course logistics
- discrete random walks

Text: Reif

Demo: strings, ball falling down peg board

Logistics

Text: *Statistical and Thermal Physics* by Reif

To be covered: most material up to Chap. 12

Grading: 15% for 10 assignments
               25% for midterm exam
               60% for final exam

Random Walk in one dimension

For the first several lectures, we discuss the random walk problem, which has widespread applicability in physics, chemistry and biology. For example
- diffusion
- configurations of polymers, including proteins and DNA
- non-interacting spins in a magnetic field

In one or two dimensions, the problem is often compared to the random motion of a drunk near a lamp-post:

origin
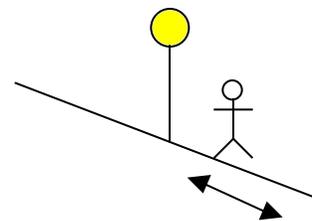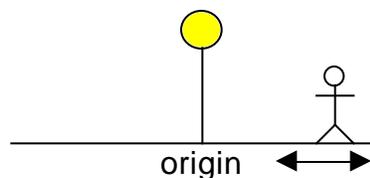
this motion can still be random, even if the probabilities are unequal

We'll take the walk to have N steps in total, of which n_R are to the right, and n_L are to the left, such that
$$N = n_L + n_R \tag{1.1}$$

We allow the probability of movement in either direction to be different:
     p = probability of move to the right
     q = probability of move to the left

where p and q must sum to unity, as the object must move somewhere
$$p + q = 1. \tag{1.2}$$

The case where $p = q = 1/2$ is special; when $p \quad q$, the situation is like a drunk on a hill.

The **essential feature** of a random walk is that the moves are **uncorrelated**: the direction taken at step i does not affect the move at $i + 1$.

For now, we assume that the steps are of equal length ℓ.

Let's consider a specific sequence with the following $N = 8$ steps:

$$p \quad\quad p \quad\quad q \quad\quad p \quad\quad p \quad\quad q \quad\quad q \quad\quad q$$

where the second line shows the individual probabilities. Here, $n_L = n_R = 4$, and the probability of this particular sequence appearing is
$$p^{n_R} q^{n_L} .$$

However, there are many different ways of arranging n_R and n_L steps, of which we have shown only one. In general, the number of ways is
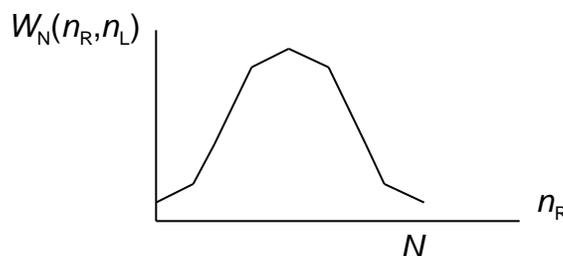$$numberof\ ways = \frac{N!}{n_R! n_L!}.$$

As a reminder, the $N!$ in the numerator arises because the number of steps is N, while the terms in the denominator are present because n_R and n_L of them are indistinguishable.

Thus, the total probability $W_N(n_R, n_L)$ of finding a given value of (n_R, n_L), including all permutations, is
$$W_N(n_R, n_L) = \frac{N!}{n_R! n_L!} p^{n_R} q^{n_L} \tag{1.3}$$

which looks like

Now, the distribution of (n_R, n_L) is referred to as the binomial distribution because it contains the same terms as are found in the binomial expansion (see Sec. 1.4):

$$(p + q)^N = \sum_{i=0}^{N} \frac{N!}{i!(N-i)!} p^i q^{N-i} \tag{1.4}$$

Comparing Eqs. (1.3) and (1.4) establishes that

$$(p + q)^N = \sum_{n_R} W_N(n_R, n_L) \tag{1.5}$$

However, $p + q = 1$, then
$$\sum_{n_R} W_N(n_R, n_L) = 1, \tag{1.6}$$

as it should be, since the probability of obtaining all the configurations is unity. This expression can be written in a variety of ways, as is explained in more detail in Reif.

Special case $p = q = 1/2$

$$W_N(n_R, n_L) = \frac{N!}{n_R! n_L!} \left(\frac{1}{2}\right)^N$$

Binomial theorem   (see Margeneau and Murphy, pp. 431-434)
In expanding $(p + q)^N$, we find the series

$$
\begin{aligned}
(p + q)^N &= p\,p\,p\ p...p & &\text{1 term, } p^N \\
&\quad + p\,p\,p\,p...p\,q + p\,p\,p\,p...q\,p + ... & &N \text{ terms } p^{N-1}q^1 \\
&\quad + ... \\
&\quad + q\,q\,q\,q...q\,q & &\text{1 term, } q^N
\end{aligned}
$$

The number of terms for a given power of p and q, taking into account distinguishability, must be
$$\frac{N!}{i!(N-i)!}$$
so that

$$(p + q)^N = \sum_{i=0}^{N} \frac{N!}{i!(N-i)!} p^i q^{N-i} .$$