Lecture 2 - Random walks - II

What's Important:

- expectations
- tip-to-tail distributions

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"Tip-to-tail" displacement

In the first lecture, we determined the probability $W_N(n_R, n_L)$ for a random walk with N steps to have n_R steps to the right and n_L steps to the left:

$$W_N(n_R, n_L) = \frac{N!}{n_R! n_L!} p^{n_R} q^{n_L} . \quad (2.1)$$

In this lecture, we show how to use this distribution to evaluate observables such as ensemble averages. In the random walk, the "tip-to-tail" distance, or equivalently the displacement of the walker, is one observable of interest. For a walk with equal length steps a, the displacement x is

$$\begin{aligned} x = ma &= (n_R - n_L)a \qquad \text{where } m = n_R - n_L \\ &= (n_R - [N - n_R])a \\ &= (2n_R - N)a. \end{aligned} \quad (2.2)$$

Eq. (2.2) demonstrates that m changes by 2 units as a function of n_R, and can range from $-N$ to $+N$.

Quick review of mean values

The function $W_N(n_R)$ tells us the probability that, in an ensemble of walks, there are walks with n_R steps to the right. From this distribution, we can extract quantities such as the average number of steps to the right, or to the left *etc*. Let's find a formal expression for the mean displacement x defined in Eq. (2.2):

$$\bar{x} = \textit{mean value of } x = \frac{\sum_i x_i \cdot [\textit{prob. of } x_i]}{\sum_i [\textit{prob. of } x_i]} \quad (2.3)$$

or

$$\bar{x} = \frac{W(0) \cdot (-N)a + W(1) \cdot (2-N)a + W(2) \cdot (4-N)a + \ldots}{W(0) + W(1) + W(2) + \ldots}$$

$$\bar{x} = \frac{\sum_i W(i) \cdot (2i - N)}{\sum_i W(i)} a \quad (2.4)$$

This is just a special case of finding the mean value of an observable

$$\bar{u} = \frac{\sum_i u_i P(i)}{\sum_i P(i)}, \quad (2.5)$$

where the denominator is frequently normalized to unity

$$\Sigma_i P(i) = 1. \quad (2.6)$$

Other quantities of interest may involve higher moments of u:

$$\overline{u^n} \; (\neq \bar{u}^n) = \frac{\sum_i u_i P(i)}{\sum_i P(i)} \quad (2.7)$$

From now on, we will assume that *P* is normalized according to Eq. (2.6).

Of particular interest is the dispersion around the mean. We define

$$\Delta u \equiv u - \bar{u} \quad (2.8)$$

where u is an observable. The mean value of Δu vanishes

$$\overline{\Delta u} = \overline{(u - \bar{u})} = \bar{u} - \overline{(\bar{u})} = 0\,,$$

but the mean value of its square does not

$$\begin{aligned} dispersion &\equiv \overline{\Delta u^2} = \overline{(u - \bar{u})^2} \\ &= \overline{(u^2 - 2u\bar{u} + \bar{u}^2)} \\ &= \overline{u^2} - 2\bar{u}^2 + \bar{u}^2 \\ &= \overline{u^2} - \bar{u}^2 \geq 0 \end{aligned} \quad (2.9)$$

The inequality follows from the left-hand side being the mean of a square, and tells us that

$$\overline{u^2} \geq \bar{u}^2 \quad (2.10)$$

Application to 1D random walks

From Eq. (2.4), we expect that

$$\bar{x} = \frac{\sum_i W(i) \cdot (2i - N)}{\sum_i W(i)} a \quad (2.11)$$

where the denominator is normalized to unity. Thus,

$$\begin{aligned} \bar{x} &= 2a \sum_{n_R} n_R W(n_R) - Na \\ &= (2\bar{n}_R - N)a \end{aligned} \quad (2.12)$$

Replacing $W_N(n_R)$ by its expression from Lec. 1 yields the formal expression

$$\bar{x} = 2a \sum_{n_R} n_R \frac{N!}{n_R!(N - n_R!)} p^{n_R} q^{n_L} - Na$$

This sum is not obvious. We evaluate it by assuming that p, q can be regarded as independent continuous variables. Then we can use the partial derivative, written as

$$n_R p^{n_R} = p \frac{\partial}{\partial p}\left(p^{n_R}\right).$$

Proof:

$$\begin{aligned}
\overline{n_R} &= \sum_{n_R} n_R \frac{N!}{n_R!(N-n_R!)} p^{n_R} q^{N-n_R} \\
&= \sum_{n_R} \frac{N!}{n_R!(N-n_R!)}\, p \frac{\partial}{\partial p}(p^{n_R})\, q^{N-n_R} \\
&= p \frac{\partial}{\partial p}\left[\sum_{n_R} \frac{N!}{n_R!(N-n_R!)} p^{n_R} q^{N-n_R} \right]
\end{aligned}$$

But the expression in the braces is just $(p+q)^N$ according to the binomial theorem. Thus

$$\begin{aligned}
\overline{n_R} &= p \frac{\partial}{\partial p}(p+q)^N \\
&= pN(p+q)^{N-1} \qquad (2.13) \\
&= Np
\end{aligned}$$

where the last line follows from $p+q=1$. This is just what we expect for p

$$p = \frac{\overline{n_R}}{N}$$

and tells us that

$$\bar{x} = (2p-1)Na. \qquad (2.14)$$

To obtain the dispersion, we apply the same derivative trick. Starting with

$$\overline{(\Delta n_R)^2} = \overline{n_R^2} - \overline{n_R}^2 = \overline{n_R^2} - (Np)^2, \qquad (2.15)$$

we have to evaluate the mean of n_R^2, which takes some work. First,

$$\begin{aligned}
\overline{n_R^2} &= \sum_{n_R} \frac{N!}{n_R!(N-n_R!)} n_R^2 p^{n_R} q^{N-n_R} \\
&= \sum_{n_R} \frac{N!}{n_R!(N-n_R!)} \left(p \frac{\partial}{\partial p}\right)^2 p^{n_R}\, q^{N-n_R} \\
&= \left(p \frac{\partial}{\partial p}\right)^2 \left[\sum_{n_R} \frac{N!}{n_R!(N-n_R!)} p^{n_R} q^{N-n_R} \right] \qquad (2.16) \\
&= \left(p \frac{\partial}{\partial p}\right)^2 (p+q)^N
\end{aligned}$$

where the second line follows from

$$\left(p \frac{\partial}{\partial p}\right)^2 p^n = p \frac{\partial}{\partial p} \cdot p \frac{\partial}{\partial p} p^n = p \frac{\partial}{\partial p} n p^n = n^2 p^n,$$

and the binomial theorem has been used to obtain the last line. Now, on to evaluate the derivatives in Eq. (2.15):

$$\begin{aligned}
\overline{n_R^2} &= p\frac{\partial}{\partial p}\left(p\frac{\partial}{\partial p}[p+q]^N\right) \\
&= p\frac{\partial}{\partial p}\left(Np[p+q]^{N-1}\right) \\
&= pN\left[(p+q)^{N-1} + p(N-1)(p+q)^{N-2}\right] \\
&= Np[1+(N-1)p] \\
&= Np(1-p+Np) \qquad \downarrow\ 1-p=q \\
&= Npq + (Np)^2
\end{aligned} \tag{2.17}$$

Now we substitute Eq. (2.17) back into (2.15) to find

$$\overline{(\Delta n_R)^2} = Npq\,. \tag{2.18}$$

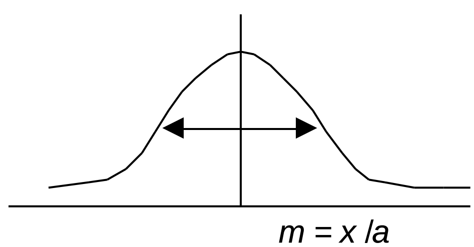
To get the dispersion in m, start with

$$\begin{aligned}
\Delta m = m - \overline{m} &= (2n_R - N) - (2\overline{n_R} - N) \\
&= 2(n_R - \overline{n_R}) \\
&= 2\Delta n_R
\end{aligned}$$

to yield

$$\overline{(\Delta m)^2} = 4\overline{(\Delta n_R)^2} = 4Npq\,. \tag{2.19}$$

Special case $p = q = 1/2$



$$\overline{(\Delta m)^2} = 4N\frac{1}{2}\frac{1}{2} = N$$

General case $p \neq q$

Here, there is a drift of the distribution away from $\langle m \rangle = 0$:

$$\begin{aligned}
\overline{m} = 2\overline{n_R} - N &= N(2p-1) = N(2p-p-q) \\
&= N(p-q)
\end{aligned}$$

The width of the distribution compared to the mean value of m is

$$\frac{\left(\overline{\Delta m^2}\right)^{1/2}}{\overline{m}} = \frac{(4Npq)^{1/2}}{N(p-q)} = \frac{1}{\sqrt{N}}\frac{2\sqrt{pq}}{(p-q)}$$

which shows that the relative width sharpens up with N.