Lecture 21 - Indistinguishable particles

What's Important:

- number distributions for MB, photons...

Text: Reif Secs. 9.4-9.7; skip 9.3

In the previous lecture, we established that the mean number density for a state ε_i could be extracted from the partition function with the aid of

$$\overline{n_i} = -\frac{1}{ß} \cdot \frac{\partial}{\partial \varepsilon_i} \ln Z \qquad (21.1)$$

Let's now use this for the various particle types, starting with Maxwell-Boltzmann.

MB number distributions

In principle, we already know the answer to this question, namely

$$\overline{n_i} = N \frac{\exp(-ß\varepsilon_i)}{\sum \exp(-ß\varepsilon_i)} \qquad (21.2)$$

although we have not worried about distinguishability before (we only examined problems with single particle states, for example a single spin). To use the partition function of the multiparticle system, we start with the Boltzmann weight

$$\exp(-ßE_R) = \exp[-ß(n_1\varepsilon_1 + n_2\varepsilon_2 + n_3\varepsilon_3 \ldots)].$$

But each state is distinguishable, that is **F** in ε_1, **M** in ε_2 (but **FM** in ε_1 is the same as **MF** in ε_1). Thus, for a given set n_1 n_2 in E_R, there are $N!\,/\,n_1!n_2!$ distinguishable states, and the sum over states becomes

$$\sum_R \rightarrow \sum_{n_1, n_2 \ldots} \frac{N!}{n_1! n_2! \ldots}$$

such that

$$Z = \sum_{n_1, n_2 \ldots} \frac{N!}{n_1! n_2! \ldots} \exp(-ßn_1\varepsilon_1 - ßn_2\varepsilon_2 \ldots)$$

$$= \sum_{n_1 n_2 \ldots} \frac{N!}{n_1! n_2! \ldots} \left(e^{-ß\varepsilon_1}\right)^{n_1} \left(e^{-ß\varepsilon_2}\right)^{n_2} \ldots$$

Now, this expression has the same form as the polynomial expansion

$$(a + b + \ldots)^N = \sum_{n_1, n_2 \ldots} \frac{N!}{n_1! n_2! \ldots} (a)^{n_1} (b)^{n_2} \ldots$$

whence

$$Z = \left(e^{-ß\varepsilon_1} + e^{-ß\varepsilon_2} + \ldots\right)^N \qquad (21.3)$$

Armed with the partition function, we can now work back to the number distribution. From Eq. (21.3),

$$\ln Z = N \ln\left(e^{-\beta\varepsilon_1} + e^{-\beta\varepsilon_2} + ...\right)$$

or

$$\ln Z = N \ln\left(\sum_r e^{-\beta\varepsilon_r}\right).$$

Invoking Eq. (21.1) for the number density

$$\bar{n}_i = -\frac{1}{\beta}\frac{\partial}{\partial\varepsilon_i} N \ln\left(\sum_r e^{-\beta\varepsilon_r}\right)$$

$$= -\frac{N}{\beta}\frac{1}{\sum_r e^{-\beta\varepsilon_r}}(-\beta)\ln\left(\sum_r e^{-\beta\varepsilon_r}\right)$$

or

$$\bar{n}_i = N\frac{\exp(-\beta\varepsilon_i)}{\sum \exp(-\beta\varepsilon_i)},$$

which is just Eq. (21.2).

Photon distributions

Photons form a Bose-Einstein system but have **no constraint on *N***, as photons can be created and destroyed through collisions. One might think that this feature makes the theoretical description more complicated, but in fact it simplifies the mathematics. Once again, we start with the partition function

$$Z = \sum_R \exp(-\beta n_1\varepsilon_1 - \beta n_2\varepsilon_2 ...)$$

The sum over R does **NOT** generate the factorials of the MB system because the particles are indistinguishable, so

$$Z = \sum_{n_1,n_2 ...} e^{-\beta n_1\varepsilon_1} e^{-\beta n_2\varepsilon_2} ...$$

$$= \sum_{n_1} e^{-\beta n_1\varepsilon_1} \sum_{n_2} e^{-\beta n_2\varepsilon_2} ... \quad (21.4)$$

Each term in the series has the same form

$$\sum_{n=0}^{\infty} e^{-\beta n\varepsilon} = 1 + \left(e^{-\beta\varepsilon}\right)^1 + \left(e^{-\beta\varepsilon}\right)^2 + ...$$

which is a geometric series

$$1 + a + a^2 + ... = \frac{1}{1-a}.$$

Thus, the partition function is

$$Z = \left(\frac{1}{1-e^{-\beta\varepsilon_1}}\right)\left(\frac{1}{1-e^{-\beta\varepsilon_2}}\right)... \qquad (21.5)$$

Now, take the logarithm and watch out for the minus sign arising from the denominators:

$$\ln Z = -\sum_r \ln\left(1-e^{-\beta\varepsilon_r}\right)$$

and invoke

$$\overline{n_i} = -\frac{1}{\beta} \cdot \frac{\partial}{\partial\varepsilon_i} \ln Z$$

for each state *r* (or *i*)

$$\overline{n_i} = -\frac{1}{\beta}\left[0 ... - \frac{\partial}{\partial\varepsilon_i}\ln\left(1-e^{-\beta\varepsilon_i}\right) + ...\right]$$

$$= +\frac{1}{\beta}\frac{(-1)(-\beta)e^{-\beta\varepsilon_i}}{\left(1-e^{-\beta\varepsilon_i}\right)}$$

$$= \frac{e^{-\beta\varepsilon_i}}{\left(1-e^{-\beta\varepsilon_i}\right)}$$

This last expression can be rearranged to yield

$$\overline{n_i} = \frac{1}{e^{\beta\varepsilon_i} - 1} \qquad (21.6)$$

This is the **Planck** distribution invoked to explain black-body radiation.

Eq. (21.6) tells us that the number density goes to zero as ε_i goes to infinity, just the usual result. But it also shows that when ε_i is small compared to $k_B T$,

$$\overline{n_i} \quad \frac{1}{1+\beta\varepsilon_i ... -1} = \frac{k_B T}{\varepsilon_i}$$

which **diverges** as ε_i vanishes. In other words, the photon distribution has the form