Lecture 22 - Bosons

What's Important:

- partition function
- number distribution
- condensation

Text: Reif

Boson partition function

We now tackle the general problem of Bose-Einstein statistics. The situation is more difficult than pure photons because the sum over n_i is now restricted by the fixed value of the particle number N. It should not surprise us to see a chemical potential appear to enforce the constraint on N.

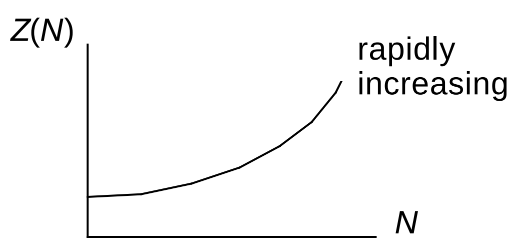
As usual, we start with the generic partition function

$$Z = \sum_R \exp(-\beta n_1 \varepsilon_1 - \beta n_2 \varepsilon_2 ...) \tag{22.1}$$

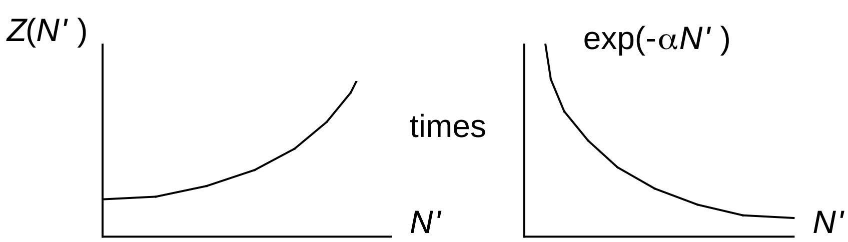
which is now subject to the constraint

$$\Sigma_i n_i = N. \tag{22.2}$$

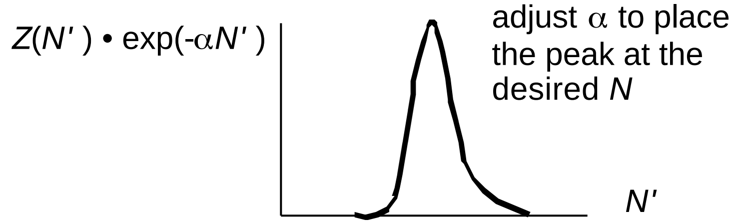
A mathematical trick is introduced to enforce the constraint. Viewing Z as a function of N, its behaviour is



Now, if we multiply the rapidly-increasing function $Z(N)$ by a rapidly-decreasing function $\exp(-\alpha N)$ we can produce a distribution with a sharp spike at the desired value of N:

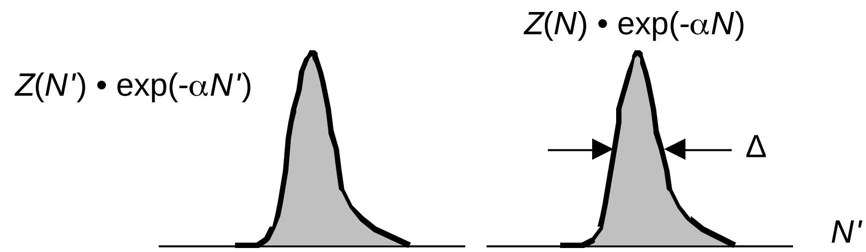


yields



In other words, the choice of N specifies the value of α to give the correct peak; α is a function of N and T.

Regarding N' as a discrete variable, the area under the curve



can be written as

$$[area] = \Sigma_{N'} \exp(-\alpha N'') Z(N'). \quad (22.3)$$

With the function so sharply peaked at $N' = N$, the area can be approximated by

$$[area] = \exp(-\alpha N) \bullet Z(N) \bullet \Delta \qquad \Delta = \textit{peak width}. \quad (22.4)$$

As will be seen in a moment, the magnitude of the peak width Δ does not appear in physical observables.

The area under the curve is defined as the *grand partition function*

$$\mathcal{Z} \equiv \Sigma_{N'} \exp(-\alpha N'') Z(N'), \quad (22.5)$$

which involves a sum over N' as well as the sum over energy states. Take the logarithm of Eq. (22.5) under the approximation of (22.4):

$$\ln\mathcal{Z} \equiv \ln[\exp(-\alpha N) \bullet Z(N) \bullet \Delta]$$

$$= -\alpha N + \ln Z(N) + \ln\Delta.$$

Now, $\ln\Delta$ is tiny compared to the other terms in this equation and can be neglected. To a good approximation then

$$\ln Z(N) = \ln\mathcal{Z} + \alpha N. \quad (22.6)$$

The advantage of working with Eq. (22.6) is that there is no restriction on the sum over *N'*, at the cost of introducing α which must be determined by the constraint.

Let's now evaluate the grand partition function:

$$\mathcal{Z} = \sum_{N'} e^{-\alpha N'} Z(N') = \sum_{R} \sum_{N'} e^{-\alpha(n_1+n_2 ...)} e^{-\alpha(\varepsilon_1 n_1 + \varepsilon_2 n_2 ...)}$$

where the sum over n_1, n_2 ... is now unrestricted. Rearranging terms leads to

$$\mathcal{Z} = \sum_{n_1, n_2 ...} e^{-n_1(\beta\varepsilon_1+\alpha)} e^{-n_2(\beta\varepsilon_2+\alpha)} ...$$

$$= \sum_{n_1=0}^{\infty} e^{-n_1(\beta\varepsilon_1+\alpha)} \sum_{n_2=0}^{\infty} e^{-n_2(\beta\varepsilon_2+\alpha)} ...$$

As before, each one of the summations in the product has the form of a geometric series, so the grand partition function can be written as

$$\mathcal{Z} = \left(\frac{1}{1-e^{-(\alpha+\beta\varepsilon_1)}}\right)\left(\frac{1}{1-e^{-(\alpha+\beta\varepsilon_2)}}\right)...$$

whence

$$\ln \mathcal{Z} = -\sum_i \ln\left(1-e^{-(\alpha+\beta\varepsilon_i)}\right)$$

This result can be placed into Eq. (22.6) to yield

$$\ln Z = \alpha N - \sum_i \ln\left(1-e^{-(\alpha+\beta\varepsilon_1)}\right) \qquad (22.7)$$

The last step is to use this to determine the number density

$$\overline{n_i} = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \varepsilon_i} = -\frac{1}{\beta}\left[\frac{-\beta e^{-(\alpha+\beta\varepsilon_i)}}{1-e^{-(\alpha+\beta\varepsilon_i)}} + \frac{\partial \ln Z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial \varepsilon_i}\right]$$

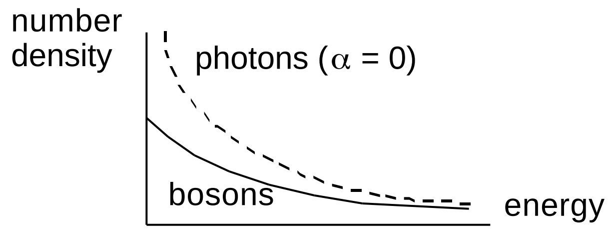
$$= \frac{e^{-(\alpha+\beta\varepsilon_i)}}{1-e^{-(\alpha+\beta\varepsilon_i)}}$$

We had to be cautious in taking the derivative because α could depend on ε_i; however, *Z* does not depend on α, so the extra terms vanish.

This is the distribution for Bose-Einstein statistics, and can be rewritten as we did for photons as

$$\overline{n_i} = \frac{1}{e^{\alpha+\beta\varepsilon_i} - 1} \qquad (22.8)$$

This distribution looks like the photon distribution except for the presence of the e^{α} term. The effect of the extra term is to remove the singularity at zero energy - the number density now remains finite, as illustrated on the following graph:



Lastly, we invoke the normalization constraint to determine α: Because

$N = \Sigma_i n_i$,

then

$$N = \sum_i \frac{1}{e^{\alpha + \beta\varepsilon_i} - 1} \qquad (22.9)$$

It is conventional to write α as $-\beta\mu$, where μ is the chemical potential.