Lecture 23 - Fermions

What's Important:
- Fermi-Dirac statistics
- comparisons among distributions
- Fermi energy

Text: Reif

Fermi-Dirac statistics

The Fermi-Dirac distribution can be handled in a similar way to the Bose-Einstein system.

Start with the partition function

$$Z = \sum_R e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + n_3\varepsilon_3 + ...)}$$

and multiply by $\exp(-\alpha N)$ to form

$$\mathcal{Z} = \Sigma_{N'} \exp(-\alpha N')\ Z(N')$$

where

$$N = n_1 + n_2 + n_3 \ ...$$

and

$$\ln\mathcal{Z} = -\alpha N + \ln Z(N) \qquad (23.1)$$

Thus

$$\mathcal{Z} = \sum_{n_1, n_2, n_3 ...} e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + n_3\varepsilon_3 + ...) - \alpha(n_1 + n_2 + n_3 + ...)}$$
$$= \left(\sum_{n_1 = 0,1} e^{-(\alpha + \beta\varepsilon_1)n_1} \right) \left(\sum_{n_2 = 0,1} e^{-(\alpha + \beta\varepsilon_2)n_2} \right) ... \qquad (23.2)$$

Because a given energy state can be occupied by only one particle at the most, each term in brackets has only two contributions, and becomes

$$1 + e^{-(\alpha + \beta\varepsilon_1)}$$

Taking the logarithm of Eq.(23.2) then gives a series

$$\ln\mathcal{Z} = \Sigma_r \ln(1 + \exp(-\alpha - \beta\varepsilon_r)) \qquad (23.3)$$

Substituting into Eq. (23.1) gives

$$\ln Z(N) = \alpha N + \Sigma_r \ln(1 + \exp(-\alpha - \beta\varepsilon_r)). \qquad (23.4)$$

Finally, with the relation

$$\bar{n}_i = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \varepsilon_i} = -\frac{1}{\beta}\frac{-\beta \exp[-(\alpha + \beta\varepsilon_i)]}{1 + \exp[-(\alpha + \beta\varepsilon_i)]}$$

we obtain

$$\bar{n}_i = \frac{1}{\exp(\alpha + \beta\varepsilon_i) + 1} \quad (23.5)$$

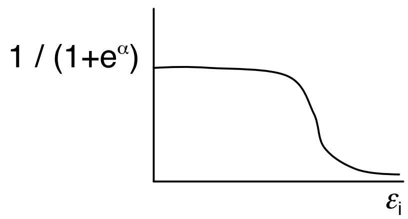
This is the **Fermi-Dirac** distribution. The constant α is fixed by the normalization condition

$$N = \sum \frac{1}{\exp(\alpha + \beta\varepsilon_i) + 1}. \quad (23.6)$$

The Fermi-Dirac distribution function has somewhat different behavior than BE or Maxwell-Boltzmann:

$\varepsilon_r \to 0$ $\quad n_r \to 1 / (e^{\alpha}+1)$

$\varepsilon_r \to \infty$ $\quad n_r \to 0$



Summary

In the last several lectures we have derived four distribution functions:

Maxwell-Boltzmann

$$\bar{n}_i = \frac{1}{\exp(\alpha + \beta\varepsilon_i)}$$

Photons

$$\bar{n}_i = \frac{1}{\exp(\beta\varepsilon_i) - 1}$$

Bose-Einstein

$$\bar{n}_i = \frac{1}{\exp(\alpha + \beta\varepsilon_i) - 1}$$

Fermi-Dirac

$$\bar{n}_i = \frac{1}{\exp(\alpha + \beta\varepsilon_i) + 1}$$

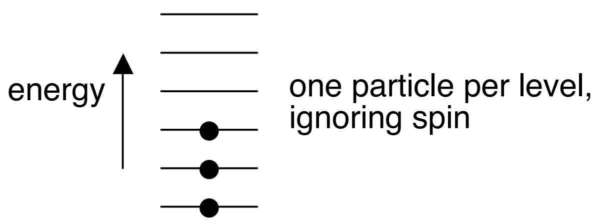
Notes:

- in the MB distribution, α is determined from $e^{-\alpha} = N / \Sigma_r \exp(-\beta\varepsilon_r)$
- for the highest energy states, all distributions tend to $\exp(-\beta\varepsilon_r)$ times a constant
- if the concentration of particles is small, or the temperature is high, then α is large, and both the BE and FD distributions tend to $\exp(-\beta\varepsilon_r)$ times a constant.

In other words, at low concentrations, or high temperatures, all distributions tend to the "classical mechanics" MB distribution.

Sign of α

Suppose that we have a FD distribution in its ground state



At $T = 0$, the distribution obeys

$$n_i = 1 \qquad \varepsilon < \varepsilon_F$$
$$n_i = 0 \qquad \varepsilon > \varepsilon_F$$

where ε_F is the Fermi energy. The constant α can be replaced by the (reduced) Fermi energy by noting that

$$\bar{n}_i = \frac{1}{\exp(\alpha + \beta\varepsilon_i) + 1}$$

has the appropriate zero temperature properties only if

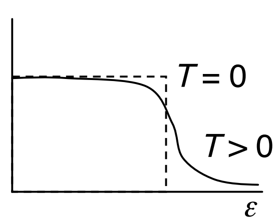
$$\alpha + \beta\varepsilon_F = 0$$

or

$$\alpha = -\beta\varepsilon_F$$

Whence, at $T = 0$ or $\beta \to \infty$:

$\varepsilon > \varepsilon_F$	$\beta(\varepsilon - \varepsilon_F) \to \infty$	and	$n = 0$
$\varepsilon > \varepsilon_F$	$\beta(\varepsilon - \varepsilon_F) \to -\infty$	and	$n = 1$
$\varepsilon = \varepsilon_F$	$\beta(\varepsilon_F - \varepsilon_F) = 0$	and	$n = 1/2$



The sign of α should not trouble us, as it was introduced only as a normalization constant.