Lecture 25 - T = 0 Fermi gas

What's Important:

- T = 0 Fermi gas
- metals
- neutron stars

Text: Reif

T=0 Fermi gas

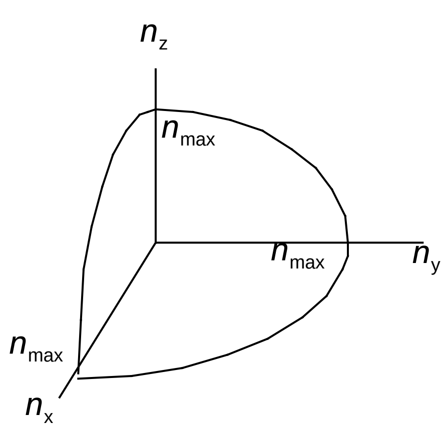
The previous lecture for the density of states in phase space dealt only with the energetics, not the statistics, of the allowed states. That is, we constructed the solutions via the "old" quantum theory and then saw how many states occupied a given volume of phase space.

Now, let's add fermions to these states. Physically, this applies to systems of electrons, protons, neutrons... individually or as multi-component systems. In three dimensions, the lowest lying states are

states	number	n_x	n_y	n_z	$n_x^2+n_y^2+n_z^2$
___ ___ ___	3	1	2	2	9
		2	1	2	9
		2	2	1	9
___ ___ ___	3	1	1	2	6
		1	2	1	6
		2	1	1	6
___	1	1	1	1	3

Putting fermions into these levels at one fermion per state takes us back to the calculation in the previous lecture. At T = 0, the particles have their lowest available energies:

- all states with $E \leq E_{max}$ are occupied
- all states with $E \geq E_{max}$ are empty



The number of fermions (of spin 1/2) allowed in these plane wave states is

$$N = 2 \bullet [\textit{spatial volume}] \bullet [4\pi p_F^3 / 3] / h^3, \qquad (25.1)$$

where the factor of two arises because there are two spin degrees of freedom associated with each spin-1/2 particle like an electron or proton.

The Fermi momentum p_F can be obtained by inverting Eq. (25.1)

$$p_F = h\,(3N / 8\pi V)^{1/3}.$$

Conducting electrons

Suppose that each atom in a metal contains one conducting electron (free to move within the material). As a rough estimate, let's take each atom to have a diameter of 2Å, so that the density of the conducting electrons is roughly

$$1 / (2 \times 10^{-10})^3 \sim 10^{29}\ \text{m}^{-3}.$$

This gives a Fermi momentum of

$$p_F = 6.626 \times 10^{-34}\,(3 \times 10^{29} / 8\pi)^{1/3} = 1.51 \times 10^{-24}\ \text{kg}\bullet\text{m/s},$$

corresponding to a Fermi energy of

$$E_F = p_F^2 / 2m = (1.51 \times 10^{-24})^2 / (2 \bullet 9.11 \times 10^{-31}) = 1.3 \times 10^{-18}\ \text{J}.$$

Converting to eV at 1 eV = 1.6×10^{-19} J gives

$$E_F \sim 8\ \text{eV},$$

a value which is pretty typical of atomic electron energies.

The speed of an electron with this Fermi momentum is

$$v_F = p_F / m = 1.51 \times 10^{-24} / 9.11 \times 10^{-31} = 1.7 \times 10^6\ \text{m/s}$$

which is just under 1% of the speed of light.

Neutron stars

A star is mainly hydrogen - protons and electrons. Consider what happens when a star is sufficiently cold and dense (i.e. uniform) that

- all electrons and protons are in their lowest available states, and
- their wavefunctions span the entire star.

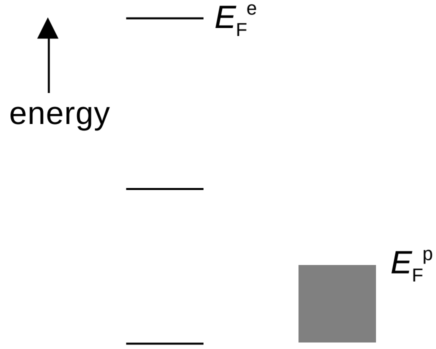
The Fermi *momentum* of all particle species with the same spatial (number) density will be the same, as

$$p_F = h\,(3N / 8\pi V)^{1/3} \qquad (25.1)$$

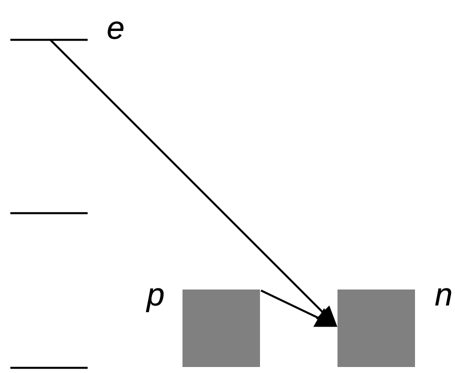
where N/V is the particle number density. However, the Fermi *energy* will vary according to the particle mass, with the *lighter* particles having the *higher* energy:

$$E_F = p_F^2 / 2m \tag{25.2}$$

In the case of protons and electrons, the Fermi energy of the electrons will be 2000 times higher than that of the protons. Thus, the energy levels will look like (can't do justice to a factor of 2000!)



When the difference between the Fermi energies exceeds the difference between the proton and neutron mass energies mc^2, an electron near E_F^e can be captured by a proton near E_F^p to create a neutron with $E \sim 0$. That is



The amount of energy required for electron capture (with the release of a massless neutrino) is

$$\Delta mc^2 = m_n c^2 - m_p c^2 - m_e c^2$$

but

$$m_n - m_p - m_e = (1.6750 - 1.6726 - 0.00091) \times 10^{-27} = 1.5 \times 10^{-30} \text{ kg}$$

so

$$\Delta mc^2 = 1.5 \times 10^{-30} \bullet (3.0 \times 10^8)^2 = 1.34 \times 10^{-13} \text{ J}.$$

As $E_F^e \sim 2000 E_F^p$, let's just equate

$$E_F^e = \Delta mc^2 = 1.34 \times 10^{-13} \text{ J}.$$

The density corresponding to this threshold for capture can then be found from inverting Eq. (25.2)

$$E_F^e = \frac{1}{2m_e} \hbar^2 \left(\frac{3N}{8\pi V}\right)^{2/3}$$

whence

$$\frac{(2m_e \Delta mc^2)^{3/2}}{\hbar^3} \cdot \frac{8\pi}{3} = \frac{N}{V}$$

and

$$\frac{N}{V} = \frac{8\pi}{3} \frac{(2 \bullet 9.11 \times 10^{-31} \cdot 1.34 \times 10^{-13})^{3/2}}{(6.626 \times 10^{-34})^3} = 3.5 \times 10^{36} m^{-3}$$

This number density (of protons or electrons) is much higher than the density of our Sun, which comes in at around 10^{30} m^{-3}. Thus, the threshold for neutron conversion in a one-solar-mass neutron star would occur at a radius about $1 / (3 \times 10^6)^{1/3} = 0.007$ that of the Sun, or

$R_{\text{threshold}} \sim 0.007 \bullet 7 \times 10^5 = 5000$ km.

The equilibrium state of a neutron star occurs when the neutron degeneracy pressure balances the gravitational pressure; at two solar masses corresponds to a radius of 10 km (see PHYS 385 lectures).