

Lecture 26 - Black-body radiation I

What's Important:

- number density
- energy density

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In our previous discussion of photons, we established that the mean number of photons with energy ϵ_i is

$$\bar{n}_i = \frac{1}{e^{\beta\epsilon_i} - 1} \tag{26.1}$$

Here, the questions that we want to address about photons are:

- what is their number density
- what is their energy density
- what is their pressure?

Number density n_γ

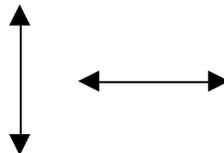
Eq. (26.1) is written in the language of discrete states. The first thing we need to do is replace the sum by an integral over photon momentum states:

$$\sum_i d^3p$$

Of course, it isn't quite this simple because the density-of-states issue that we introduced before: for every spin state there is one phase space state for every h^3 , so the proper replacement more like

$$\sum_i (1/h^3) d^3p d^3r$$

The units now work: the left hand side is a number, and so is the right hand side. But this expression ignores spin - it just deals with the states in phase space. For every photon momentum, there are two ways of arranging its polarization (*i.e.*, the orientation of its electric or magnetic field vectors):



where the photon momentum vector is perpendicular to the plane. Thus, we have

$$\sum_i (2/h^3) d^3p d^3r. \tag{26.2}$$

Assuming that our system is spatially uniform, the position integral can be replaced by the volume

$$d^3r = V.$$

The total number of photons is then

$$N = 2 \frac{V}{h^3} \frac{d^3 p}{e^{\beta \epsilon} - 1}. \quad (26.3)$$

This integral is written in terms of energy ϵ and momentum p ; for photons, these quantities are related through Einstein's relation ($m = 0$)

$$\epsilon = pc.$$

Dividing N by V gives the integrated number density of photons n , which can be written in an integral form as

$$n_\gamma = \frac{2}{h^3} \frac{d^3 p}{e^{\beta pc} - 1} \quad (26.4)$$

or as

$$[\text{number per unit volume between } \mathbf{p} \text{ and } \mathbf{p}+d\mathbf{p}] = n(\mathbf{p}) d^3 p$$

with

$$n_\gamma(\mathbf{p}) d^3 p = \frac{2}{h^3} \frac{d^3 p}{e^{\beta pc} - 1}. \quad (26.5)$$

Let's integrate Eq. (26.4) for an isotropic system, where $\int_{\text{angles}} d^3 p = 4 \int p^2 dp$:

$$n_\gamma = \frac{8}{h^3} \int_0^\infty \frac{p^2 dp}{e^{\beta pc} - 1}.$$

Changing variables to $x = \beta pc$ gives

$$n_\gamma = \frac{8}{(\beta hc)^3} \int_0^\infty \frac{x^2 dx}{e^x - 1}$$

The integral has the form of a Reimann Zeta function, and is

$$\int_0^\infty \frac{x^2 dx}{e^x - 1} = 2\zeta(3) = 2 \cdot 1.202$$

Thus,

$$n_\gamma = 1.202 \cdot 16 \frac{k_B T}{hc}^3 \quad (26.6)$$

Example The universe is filled with relic radiation left over from the Big Bang with a temperature of 2.7 K. What is n of this radiation?

$$\begin{aligned} n_\gamma &= 1.202 \cdot 16 \frac{1.38 \times 10^{-23} \cdot 2.7}{6.63 \times 10^{-34} \cdot 3.0 \times 10^8}^3 \\ &= 4.0 \times 10^8 \text{ m}^{-3} = 400 \text{ cm}^{-3} \end{aligned}$$

Energy density u_γ

We saw that the number of photons per unit volume between \mathbf{p} and $\mathbf{p}+d\mathbf{p}$ is

$$n_\gamma(\mathbf{p})d^3p = \frac{2}{h^3} \frac{d^3p}{e^{\beta pc} - 1}.$$

These photons have an energy of pc , giving an energy density $u(\mathbf{p})d^3p$ in this momentum range of

$$u_\gamma(\mathbf{p})d^3p = pc \frac{2}{h^3} \frac{d^3p}{e^{\beta pc} - 1}.$$

Taking the integral of this over momentum gives the total energy density u for an isotropic system,

$$u_\gamma = \frac{8}{h^3} pc \frac{p^2 dp}{e^{\beta pc} - 1} = \frac{8c}{h^3} \frac{p^3 dp}{e^{\beta pc} - 1}.$$

where $\int_{\text{angles}} d^3p = 4\pi p^2 dp$ as before. Once again changing variables to $x = \beta pc$

$$u_\gamma = \frac{8c}{h^3} \frac{kT}{c} \int_0^\infty \frac{x^3 dx}{e^x - 1} = 8 hc \frac{kT}{hc} \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

The integral is equal to

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}.$$

Thus,

$$u_\gamma = \frac{8}{15} \frac{(k_B T)^4}{(hc)^3}. \quad (26.7)$$

Example Find the energy density of photons at room temperature $T = 293$ K.

$$u_\gamma = \frac{8}{15} \frac{(1.38 \times 10^{-23} \cdot 293)^4}{(6.63 \times 10^{-34} \cdot 3.0 \times 10^8)^3} = 5.5 \times 10^{-6} \text{ J/m}^3.$$

Mean energy per photon

From Eqs. (26.6) and (26.7) we can determine the mean energy per photon

$$\frac{u_\gamma}{n_\gamma} = \frac{\frac{8}{15} \cdot \frac{(k_B T)^4}{(hc)^3}}{1.202 \cdot 16 \frac{(k_B T)^3}{(hc)^3}} = \frac{4}{1.202 \cdot 30} k_B T = 2.7 k_B T \quad (26.8)$$

Notice once again how $k_B T$ sets the energy scale.