Lecture 27 - Black-body radiation II

What's Important:

- radiation pressure
- radiative power

Text: Reif

Radiation pressure

The general definition of the mean force ϕ for a change in its conjugate variable h is

$$\beta\phi = \frac{\partial \ln Z}{\partial h} \tag{27.1}$$

which for the mean pressure implies

$$\beta\overline{p} = \frac{\partial \ln Z}{\partial V}. \tag{27.2}$$

For a photon gas, we determined that

$$\ln Z = -\sum_i \ln\left(1 - e^{-\beta\varepsilon_i}\right)$$

so Eq. (27.2) implies

$$\begin{aligned}
\beta\overline{p}_\gamma &= \frac{\partial}{\partial V}\left[-\sum_i \ln\left(1 - e^{-\beta\varepsilon_i}\right)\right] \\
&= -\sum_i \left(\frac{\partial \varepsilon_i}{\partial V}\right)\frac{\partial}{\partial \varepsilon_i}\ln\left(1 - e^{-\beta\varepsilon_i}\right) \\
&= -\sum_i \left(\frac{\partial \varepsilon_i}{\partial V}\right)(-1)\frac{(-\beta\varepsilon_i)e^{-\beta\varepsilon_i}}{\left(1 - e^{-\beta\varepsilon_i}\right)} \\
&= -\beta\sum_i \left(\frac{\partial \varepsilon_i}{\partial V}\right)\frac{\varepsilon_i e^{-\beta\varepsilon_i}}{\left(1 - e^{-\beta\varepsilon_i}\right)}
\end{aligned} \tag{27.3}$$

Now, how does a photon energy change with volume? Consider a specific quantum state n_i in a cubic cavity of volume $V = L^3$. From the usual standing wave condition, the wavelength λ_i is

$$\lambda_i = 2L/n_i$$

so

$$p_i = h/\lambda_i = h/(2L/n_i) = (h/2L)n_i,$$

and the energy is

$$\varepsilon_i = p_i c = (hc/2L)n_i = (hcn_i/2V^{1/3}) = (hcn_i/2)V^{-1/3}.$$

The derivative is then

$$
\begin{aligned}
\frac{\partial \varepsilon_i}{\partial V} &= \frac{\partial}{\partial V}\left[\left(\frac{hcn_i}{2}\right)V^{-1/3}\right] = \left(\frac{hcn_i}{2}\right)\left(\frac{-1}{3}\right)V^{-4/3} \\
&= -\frac{1}{3}\frac{1}{V}\left(\frac{hcn_i}{2}\right)V^{-1/3} \qquad (27.4)\\
&= -\frac{1}{3V}\varepsilon_i
\end{aligned}
$$

Placing Eq. (27.4) into (27.3) gives

$$
\beta \bar{p}_\gamma = (-1)^2 \frac{\beta}{3V} \sum_i \frac{\varepsilon_i e^{-\beta\varepsilon_i}}{\left(1-e^{-\beta\varepsilon_i}\right)} = \frac{\beta}{3V}\bar{\varepsilon}_\gamma \qquad (27.5)
$$

where $\bar{\varepsilon}_\gamma$ is the mean photon energy. Now, the energy density is just

$$
u_\gamma = \bar{\varepsilon}_\gamma / V
$$

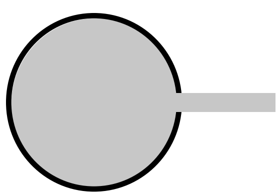
permitting us to rewrite (27.5) as

$$
\bar{p}_\gamma = \frac{1}{3} u_\gamma \,.
$$

In other words, the radiation pressure is equal to one-third of the energy density.

Radiant power

In black-body radiation, the surface is considered to be ideal, emitting and absorbing photons with 100% efficiency. To determine the black-body radiation rate, we examine emission from a cavity containing a photon gas in equilibrium:



photons emitted from
a hole in a cavity

In a previous lecture, we commented that effusion of an ideal gas from a container was proportional to the flux of particles, not the equilibrium distribution itself. The same applies here, so our first task is to calculate the photon flux:

[*flux in range p to p+dp and* Ω *to* $\Omega+d\Omega$]

= [*number per unit area per unit time in range p to p+dp and* Ω *to* $\Omega+d\Omega$]

= $c \cos\theta \, n_\gamma(\mathbf{p}) \, p^2 dp \, d\Omega$,

where the $c \cos\theta$ factor arises like the $v \cos\theta$ in the ideal gas problem.

Multiplying the flux by the energy per photon pc gives the power per unit area P_γ, after integrating over d^3p:

$$P_\gamma = \int d\Omega \; pcc \cos\theta \; n_\gamma(\mathbf{p}) p^2 dp$$
$$= 2\pi c \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int n_\gamma(\mathbf{p}) cp^3 dp$$

The angular integral just gives a factor of 1/2 (because only a hemisphere contributes)

$$\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = \int_0^1 \cos\theta \, d\cos\theta = \frac{1}{2},$$

while the momentum integral gives the energy density divided by 4π

$$\int n_\gamma(\mathbf{p}) cp^3 dp = \int \varepsilon \, n_\gamma(\mathbf{p}) p^2 dp = \frac{u_\gamma}{4\pi}.$$

Thus,

$$P_\gamma = 2\pi c \cdot \frac{1}{2} \cdot \frac{u_\gamma}{4\pi} = \frac{c}{4} u_\gamma \,. \qquad (27.6)$$

We showed before that the energy density is given by

$$u_\gamma = \frac{8\pi}{15} \frac{(\pi k_B T)^4}{(hc)^3} = \frac{\pi}{30} \frac{(2\pi k_B T)^4}{(hc)^3}$$

so

$$P_\gamma = \frac{c}{4} \cdot \frac{\pi}{30} \cdot \frac{(2\pi k_B T)^4}{(hc)^3}$$
$$= \left(\frac{2\pi^5 k_B^4}{15 c^2 h^3} \right) T^4 \qquad (27.7)$$
$$\equiv \sigma T^4$$

This is known as the Stephan-Boltzmann Law, and σ is called the Stephan-Boltzmann constant. Numerically, σ is

$$\sigma = \frac{2\pi^5 k_B^4}{15 c^2 h^3} = \frac{2\pi^5 (1.38 \times 10^{-23})^4}{15 \cdot (3.0 \times 10^8)^2 \cdot (6.63 \times 10^{-34})^3} \qquad (27.8)$$
$$= 5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$$

Example If the Sun has a surface temperature of 5600 K and a radius of 6.96×10^8 meters, what is its total power output (or *luminosity*)?

$$[\textit{luminosity}] = 4\pi R^2 P_\gamma = 4\pi \, (6.96 \times 10^8)^2 \, 5.67 \times 10^{-8} \, (5600)^4 = 3.4 \times 10^{26} \text{ watts.}$$