

## Lecture 28 - Superfluids

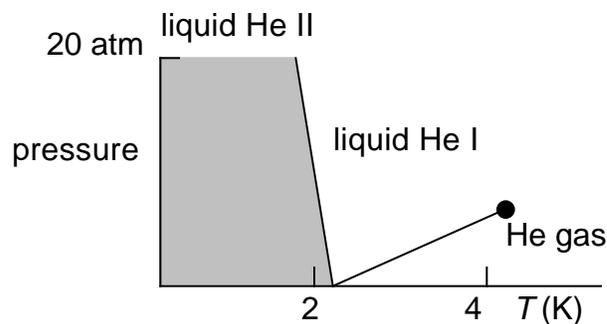
*What's Important:*

- $^4\text{He}$  condensation
- superfluids
- Bose-Einstein condensation

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 **$^4\text{He}$  condensation**

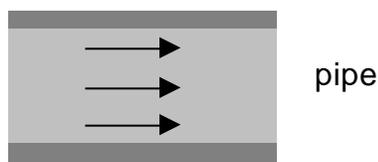
As  $^4\text{He}$  is cooled, it undergoes a phase transition to a liquid state (referred to as He I) at  $T = 4.2\text{ K}$ . Upon further cooling (less than  $2.2\text{ K}$  at  $1\text{ atm}$  pressure), He I changes to He II, also a liquid.



He II is not a normal liquid, and is referred to as a superfluid.

Superfluids are characterized by

- high heat conductivity
- superleaks (can flow through holes  $100\text{ \AA}$  across)
- low viscosity (no gradient across fluid when it flows through a pipe)



Such behavior is collective, in that there is little relative motion. It is thought that superfluid He II is a Bose-Einstein condensate in which most atoms are in their ground state so that they move cooperatively.

**Bose-Einstein condensates**

Consider the behavior of a single-component Bose-Einstein gas with no interactions between particles - just statistics. From a previous lecture (22), the mean number density obeys

$$\bar{n}_i = \frac{1}{e^{\alpha + \beta \varepsilon_i} - 1} \quad (28.1)$$

Since we assume ideal particles with no interactions and no internal structure, the energies are the usual

$$\varepsilon_i = p_i^2 / 2m$$

associated with kinetic energy. Hence,  $\varepsilon_i$  ranges from 0 to  $\infty$ .

To find the total number of particles  $N$ , we just perform the sum

$$N = \sum_i \bar{n}_i = \sum_i \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1}$$

over all  $n_i$ . If the states are close enough in energy as to be regarded as continuous, then the summation can be replaced by the integral

$$\sum_i \frac{V}{h^3} d^3p = \frac{4}{h^3} V \int p^2 dp$$

such that

$$N = \frac{4}{h^3} V \int \frac{p^2 dp}{e^{\beta(\varepsilon - \mu)} - 1}$$

where  $\varepsilon = p^2 / 2m$  as above.

Now, if  $\mu$  has any positive value, then there will be some range of  $\varepsilon$  where

- the value of  $\varepsilon - \mu$  is negative, so that
- $\exp(\beta[\varepsilon - \mu])$  is less than one, so that
- $\{\exp(\beta[\varepsilon - \mu]) - 1\}$  is negative!

This is not allowed, so we conclude that

**$\mu$  must be negative.**

Now, we evaluate the integral for negative  $\mu$ . Start with the usual change of variables

$$(\beta / 2m)^{1/2} p = x$$

so

$$\frac{N}{V} = \frac{4}{h^3} \frac{2m}{\beta}^{3/2} \frac{x^2 dx}{e^{x^2} e^{-\beta \mu} - 1}. \quad (28.2)$$

Some more manipulations:

- define the fugacity  $z = \exp(+\beta\mu)$ , which is less than unity for  $\mu < 0$
- change variables to  $y = x^2$ , with  $dy = 2x dx$

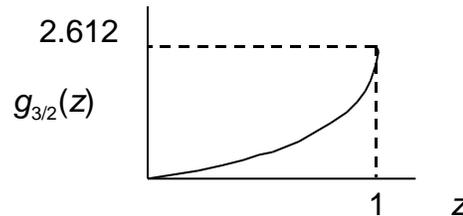
leading to

$$\frac{N}{V} = 2 \frac{2mk_B T}{h^2}^{3/2} \frac{y^{1/2} dy}{e^y z^{-1} - 1}. \quad (28.3)$$

Now, the integral is proportional to the Bose-Einstein function, which depends on  $\mu$  through  $z$

$$\frac{y^{1/2} dy}{e^y z^{-1} - 1} = \frac{\sqrt{z}}{2} g_{3/2}(z). \tag{28.4}$$

Over the allowable range  $0 < z < 1$  the function  $g_{3/2}(z)$  varies from 0 to 2.612 (which is the value of the Reimann zeta function  $\zeta(3/2)$ ). Schematically:



In other words, upper bound on the Bose-Einstein function sets an upper bound on the the right hand side of Eq. (28.3)

$$\frac{N}{V} = 2.612 \cdot 2 \frac{2mk_B T}{h^2}^{3/2}. \tag{28.5}$$

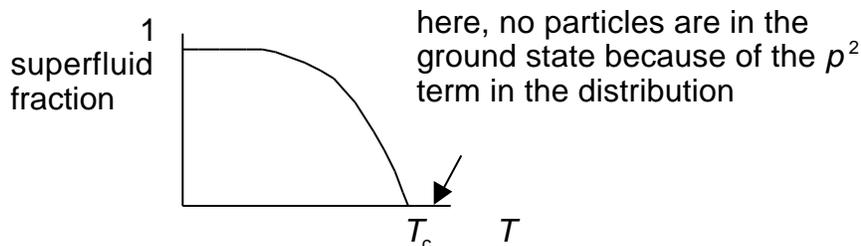
What does this tell us?

1. For a given density, there is a minimum temperature in order for all the particles in the system to obey the BE equilibrium distribution.
2. If the temperature falls below this value, then there is "extra density", and these particles must go into the  $p = 0$  state (they are then removed from the equation for  $N/V$  because their number is multiplied by  $p^2$ ).

We call the threshold temperature the critical temperature  $T_c$ . The particles at  $p = 0$  are the superfluid state. Thus,

$$[\text{superfluid density}] = [\text{total density}] - [\text{normal density from } g_{3/2}]$$

Schematically, we expect



*Example* Find  $T_c$  for a density of  $2 \times 10^{28}$  atoms/m<sup>3</sup>, which is the density of helium if its mass density is 0.145 g/cm<sup>3</sup>.

Substituting into Eq. (28.5) gives  $T_c = 2.9$  K, which is not bad.