Lecture 3 - Random walks at large *N*

What's Important:
•   random walks in the continuum limit
Text: Reif

1D Random walks at large *N*

As the number of steps *N* in a walk becomes very large, the individual step sizes in n_R become relatively small. Further, the difference in values of $W(n_R)$ between successive values of n_R becomes small as well. The proof of how the difference scales with *N* is given in Reif. For our purposes, the important point is that we can regard n_R and $W(n_R)$ as continuous at large *N*.

To obtain a continuous description of $W(n_R)$, consider its form near its most likely value at $\tilde{n}_R$
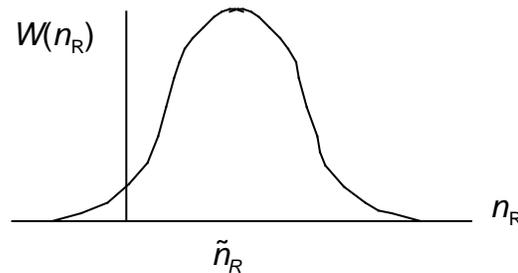
At the peak of the distribution, its derivative with respect to $\tilde{n}_R$ vanishes

$$\left.\frac{dW}{dn_R}\right|_{\tilde{n}_R} = 0 \quad \text{or} \quad \left.\frac{d\ln W}{dn_R}\right|_{\tilde{n}_R} = 0$$

Let's expand $W(n_R)$ or its logarithm around $\tilde{n}_R$ using η as an expansion parameter, where

$$n_R = \tilde{n}_R + \eta.$$

Choosing the logarithm, for example, we have

$$\ln W(n_R) = \ln W(\tilde{n}_R) + B_1\eta + \tfrac{1}{2}B_2\eta^2 + ... \tag{3.1}$$

where

$$B_i = \left.\frac{d^i \ln W}{dn_R^i}\right|_{\tilde{n}_R}$$

At the maximum in *W*, the coefficient $B_1 = 0$ by definition. The coefficients B_i can be obtained from the discrete expression for *W* in terms of factorials. Start with the definition

$$\ln W(n_R) = \ln N! - \ln n_R! - \ln(N-n_R)! + n_R\ln p + (N-n_R)\ln q,$$

so

$$\frac{d \ln W(n_R)}{dn_R} = 0 - \frac{d}{dn_R} \ln n_R! - \frac{d}{dn_R} \ln(N - n_R)! + \ln p - \ln q. \tag{3.2}$$

Now, the derivatives of the factorials can be approximated by

$$\frac{d \ln n!}{dn} \cong \frac{\ln(n+1)! - \ln n!}{1} = \ln \frac{(n+1)!}{n!} = \ln(n+1) \cong \ln n \tag{3.3}$$

Thus, Eq. (3.2) becomes

$$\frac{d \ln W(n_R)}{dn_R} = -\ln n_R + \ln(N - n_R) + \ln \frac{p}{q} \tag{3.4}$$

where care has been taken with the sign of the derivative of $\ln(N\text{-}n_R)!$

Eq. (3.4) gives an expression for B_1, which can be used to obtain

$$\tilde{n}_R = Np \tag{3.5}$$

by setting $B_1 = 0$. Of course, we already know Eq. (3.5) from general considerations of the random walk.

Pressing on to extract B_2, we first take another derivative of W:

$$\frac{d^2 \ln W(n_R)}{dn_R{}^2} = \frac{d}{dn_R}\left[-\ln n_R + \ln(N - n_R) + \ln \frac{p}{q} \right]$$

$$= -\frac{1}{n_R} + (-1)\frac{1}{N - n_R} + 0 \tag{3.6}$$

$$= -\frac{1}{n_R} - \frac{1}{N - n_R}$$

We evaluate this at $n_R = \tilde{n}_R = Np$ to determine B_2

$$B_2 = \frac{d^2 \ln W(n_R)}{dn_R{}^2}\bigg|_{\tilde{n}_R} = -\frac{1}{\tilde{n}_R} - \frac{1}{N - \tilde{n}_R}$$

$$= -\frac{1}{Np} - \frac{1}{N - Np}$$

$$= -\frac{1}{N}\left[\frac{1}{p} + \frac{1}{1-p} \right] \tag{3.7}$$

$$= -\frac{1}{N} \cdot \frac{1-p+p}{p(1-p)}$$

$$= -\frac{1}{N} \cdot \frac{1}{pq}$$

Higher-order terms in B are smaller by successive powers of N and are not important for our level of approximation.

Thus, inserting B_1 and B_2 into the expansion (3.1) gives

$$\ln W(n_R) = \ln W(\tilde{n}_R) + 0 + \frac{1}{2}\frac{1}{Npq}\eta^2 + ...$$

With the definition

$$\tilde{W} \equiv W(\tilde{n}_R)$$

this expression yields

$$W(n_R) = \tilde{W}e^{-\eta^2/2Npq} \tag{3.8}$$

The implications of this Gaussian form will be discussed momentarily. But first, we obtain an expression for the prefactor by means of the normalization condition

$$\int W(\tilde{n}_R + \eta)d\eta = 1$$

where we assume that N is so large that the integration limits are effectively infinite. Substituting Eq. (3.8) for $W(n_R)$ gives

$$1 = \tilde{W}\int e^{-\eta^2/2Npq}d\eta$$

$$= \tilde{W}(2Npq)^{1/2}\int e^{-x^2}dx$$

$$= \tilde{W}(2Npq)^{1/2}\sqrt{}$$

Inverting

$$\tilde{W} = (2\ Npq)^{-1/2} \tag{3.9}$$

The proof for the value of the integral is

Define $C = \int \exp(-y^2)dy$ so that

$$C^2 = \int \exp(-x^2)dx \int \exp(-y^2)dy$$

$$= \int \exp[-(x^2+y^2)]dx\,dy$$

$$= \int_0^2 d\theta \int_0 \exp(-r^2)\,r\,dr$$

$$= 2\ (1/2)\int_o \exp(-z)\,dz \quad (z = r^2)$$

$$= .$$

Thus, $C = \quad .$

At long last, we have the complete form, after substituting $\eta = n_R - \tilde{n}_R = n_R - Np$:

$$W(n_R) = \frac{1}{(2\ Npq)^{1/2}}\exp\ -\frac{(n_R - Np)^2}{2Npq} \tag{3.10}$$

An alternate form is to write this in terms of n_R and its dispersion

$$\overline{n_R^2} = Npq$$

so that

$$W(n_R) = \frac{1}{(2\pi \overline{n_R^2})^{1/2}} \exp\left(-\frac{(n_R - \overline{n_R})^2}{2\overline{n_R^2}}\right) \qquad (3.11)$$
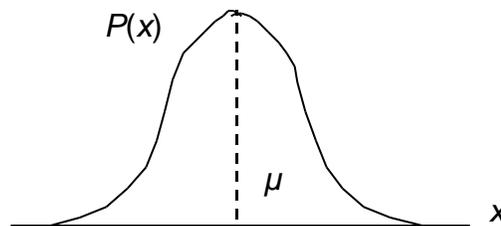
This distribution is of a Gaussian form, peaked at its mean value. We've done this for a *number*, which is a special case of the Gaussian probability *density*

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad (3.12)$$

defined such that the probability of a variable x having a value between x and $x + dx$ is $P(x)\, dx$.

Identifying parameters in the distributions leads to

$$\sigma^2 = \overline{n_R^2} \qquad\qquad \mu = \overline{n_R}. \qquad (3.13)$$

Note that the full width at half maximum **IS NOT 2σ but rather 2.355σ.**

The distribution in displacements is also Gaussian.   To find $P(m)$, which is the probability of m steps to the right, use

$$N = n_R + n_L \qquad\text{and}\qquad m = n_R - n_L$$

to give

$$n_R = (N + m) / 2.$$

Thus,

$$P(m) = W(n_R = [N+m]/2) = \frac{1}{(2\pi Npq)^{1/2}} \exp\left(-\frac{([N+m]/2 - Np)^2}{2Npq}\right)$$

$$= \frac{1}{(2\pi Npq)^{1/2}} \exp\left(-\frac{(m - N[p-q])^2}{8Npq}\right) \qquad (3.14)$$

Now, $P(m)$ is the probability of finding a given value of m in an ensemble of walks with N steps. If we wish m to be continuous, rather than discrete, then we must work with probability densities:

[*probability of finding y in the range y to y + dy*]
        = [*probability density at y*] • [*range dy*]

or

[*probability of finding y in the range y to y + dy*] = $P(y)\,dy$.

As applied to random walks, this becomes

$P(n_R)\,dn_R = W(n_R)\,dn_R = (1/2)W(n_R)\,dm$

where the last equality arises from

$$\frac{dm}{dn_R} = \frac{d}{dn_R}(2n_R - N) = 2$$

In other words, because *m* increases by 2 units for every increase in n_R by one unit:

[*probability of finding n_R between n_r and n_R + 1*]
    = [*probability of finding m between m and m + 2*]

The same argument can be repeated to give probabilities as a function of *x*, which has dimensions of length. Let the step size be *l*, so

[*probability of finding n_R between n_R and n_R + 1*]
    = [*probability of finding x between x and x + 2l*]

or

$P(x)\,dx = [W(n_R) / 2l]\,dx.$ (3.15)