Lecture 30 - Low temperature Fermi gas

What's Important:
• heat capacity for ideal Fermi gas
Text: Reif

Heat capacity for ideal Fermi gas

This lecture contains a somewhat lengthy mathematical examination of non-interacting Fermi gases at low temperatures. First, the mean energy $\overline{E}$ of the gas is determined, which permits the calculation of the heat capacity C_v via

$$C_v = \frac{\partial \overline{E}}{\partial T}$$ (30.1)

Mean energy

As shown in previous lectures, a gas of fermions without interactions obeys the number distribution

$$\overline{n}_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} + 1},$$ (30.2)

where μ is alternatively the Fermi energy ε_F. Converting this to a continuum distribution gives the mean number of electrons in the momentum range **p** to **p** + d**p** as

$$\overline{n}(p)d^3p = 2\frac{V}{h^3}\frac{d^3p}{e^{\beta(\varepsilon_i - \mu)} + 1}$$

where the factor of two comes from the spin-1/2 nature of the electrons. For the problem at hand, it is more useful to know the distribution of electron energies, rather than momenta, and so we convert using

$$\varepsilon = p^2/2m \qquad\qquad \frac{d\varepsilon}{dp} = \frac{1}{2m} \bullet 2p = \frac{p}{m} = \frac{\sqrt{2m\varepsilon}}{m}$$

or

$$dp = \frac{m}{\sqrt{2m\varepsilon}}\,d\varepsilon.$$ (30.3)

Integrating over angles with $\int_{angles}d^3p = 4\pi p^2 dp$, Eq. (30.2) becomes

[*number of electrons between ε and ε+dε*] =

$$2\frac{V}{h^3}4\pi p^2 \frac{dp}{e^{\beta(\varepsilon - \mu)} + 1}$$

$$= 2\frac{V}{h^3}4\pi(2m\varepsilon)\frac{m}{\sqrt{2m\varepsilon}}\frac{d\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1}$$ (30.4)

$$D(\varepsilon)\frac{d\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1}$$

The density of states factor $D(\varepsilon)$ is given by

$$D(\varepsilon) = \frac{\sqrt{2}}{2} V \left(\frac{m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2} \, . \tag{30.5}$$
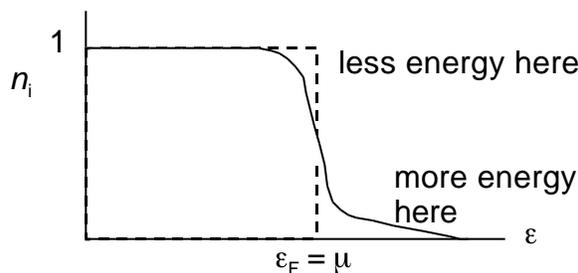
Just to make the notation a little easier, define

$$f(\varepsilon) \equiv \frac{1}{e^{\beta(\varepsilon - \mu)} + 1} \, , \tag{30.6}$$

which permits Eq. (30.4) to be written as

[*number of electrons between ε and $\varepsilon + d\varepsilon$*] $= f(\varepsilon)D(\varepsilon)d\varepsilon$. $\hphantom{xxxxxx}$ (30.7)

Compared to the ground state distribution at $T = 0$, how much energy E does a Fermi gas have at $T > 0$?

At $T = 0$, the energy is

$$E(T=0) = \varepsilon_F N$$

where

$$N = \int_o f(\varepsilon)D(\varepsilon)d\varepsilon.$$

At $T > 0$, this changes to

$$E = \int_o \varepsilon \, f(\varepsilon)D(\varepsilon)d\varepsilon.$$

Thus

$$E = \int_o \varepsilon \, f(\varepsilon)D(\varepsilon)d\varepsilon - \varepsilon_F \int_o f(\varepsilon)D(\varepsilon)d\varepsilon$$
$$= \int_o (\varepsilon - \varepsilon_F) \, f(\varepsilon)D(\varepsilon)d\varepsilon. \tag{30.8}$$

Calculation of specific heat

Knowing the energy change, we can now evaluate the specific heat through

$$C_v = \frac{\partial \bar{E}}{\partial T} \, .$$

Now, the only temperature dependent term in Eq. (30.8) is $f(\varepsilon)$, so the derivative is

$$C_v = \frac{\partial E}{\partial T} = \int_0 (\varepsilon - \varepsilon_F) \frac{\partial f(\varepsilon)}{\partial T} D(\varepsilon) d\varepsilon \, . \tag{30.9}$$

Over most of the integration range in Eq. (30.9), $D(\varepsilon)$ is a slowly varying function, because $D(\varepsilon) \sim \varepsilon^{1/2}$. Therefore, we can take $D(\varepsilon)$ out of the integral and evaluate it at the place where the integrand has its largest value, namely ε_F, leaving

$$C_v = D(\varepsilon_F) \int_0 (\varepsilon - \varepsilon_F) \frac{\partial (e^{\beta(\varepsilon - \varepsilon_F)} + 1)^{-1}}{\partial T} d\varepsilon$$

$$= D(\varepsilon_F) \int_0 (\varepsilon - \varepsilon_F)(-1)(e^{\beta(\varepsilon - \varepsilon_F)} + 1)^{-2} \frac{\partial e^{\beta(\varepsilon - \varepsilon_F)}}{\partial T} d\varepsilon$$

The remaining derivative is straightforward:

$$\frac{\partial e^{\beta(\varepsilon - \varepsilon_F)}}{\partial T} = -(\varepsilon - \varepsilon_F) \frac{1}{k_B T^2} e^{\beta(\varepsilon - \varepsilon_F)},$$

and leads to

$$C_v = D(\varepsilon_F) \int_0 (\varepsilon - \varepsilon_F)(-1) \left(e^{\beta(\varepsilon - \varepsilon_F)} + 1 \right)^{-2} (-1)(\varepsilon - \varepsilon_F) \frac{1}{k_B T^2} e^{\beta(\varepsilon - \varepsilon_F)} d\varepsilon$$

or

$$C_v = D(\varepsilon_F) \frac{1}{k_B T^2} \int_0 (\varepsilon - \varepsilon_F)^2 \frac{e^{\beta(\varepsilon - \varepsilon_F)}}{\left(e^{\beta(\varepsilon - \varepsilon_F)} + 1 \right)^2} d\varepsilon .$$
(30.10)

Next, we change variables to the dimensionless quantity $x = \beta(\varepsilon - \varepsilon_F)$

$$C_v = D(\varepsilon_F) \frac{1}{k_B T^2} (k_B T)^3 \int_{-\beta\varepsilon_F} x^2 \frac{e^x}{\left(e^x + 1 \right)^2} dx$$

$$= D(\varepsilon_F) k_B^2 T \int_{-\beta\varepsilon_F} x^2 \frac{e^x}{\left(e^x + 1 \right)^2} dx$$

At low temperatures, $\beta\varepsilon_F$ is large, and the lower limit of the integral can be taken to - . The resulting integral has the exact value (see Reif)

$$\int_- x^2 \frac{e^x}{\left(e^x + 1 \right)^2} dx = \frac{2}{3}$$

This, together with the expression for $D(\varepsilon_F)$, gives

$$C_v = \frac{\sqrt{2}}{2} \left(\frac{m}{\hbar^2} \right)^{3/2} \varepsilon_F^{1/2} k_B^2 T V \frac{2}{3}$$

$$= \frac{\sqrt{2}}{3} \left(\frac{m}{\hbar^2} \right)^{3/2} \varepsilon_F^{1/2} V k_B^2 T$$
(30.11)

$$= \frac{1}{6} \left(\frac{2m}{\hbar^2} \right)^{3/2} \varepsilon_F^{1/2} V k_B^2 T$$

The important result here is that C_v is linear in temperature, as is observed experimentally. To get rid of the V term in Eq. (30.11), we invert the expression for the Fermi energy

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}$$

$$V\left(\frac{2m}{\hbar^2}\right)^{3/2} = \frac{3\pi^2 N}{\varepsilon_F^{3/2}}$$

such that

$$C_v = \frac{\pi^2}{2}Nk_B\frac{k_B}{\varepsilon_F}T$$

$$C_v = \frac{\pi^2}{2}Nk_B\frac{T}{T_F}$$