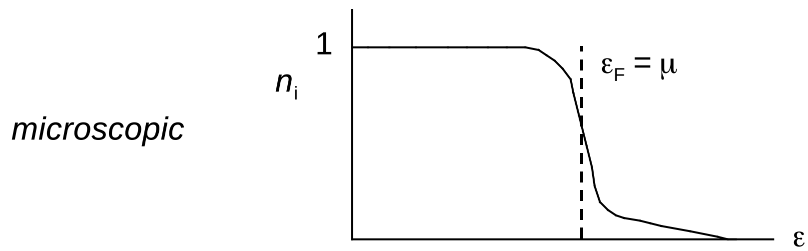
Lecture 31 - Interactions: Phase transitions

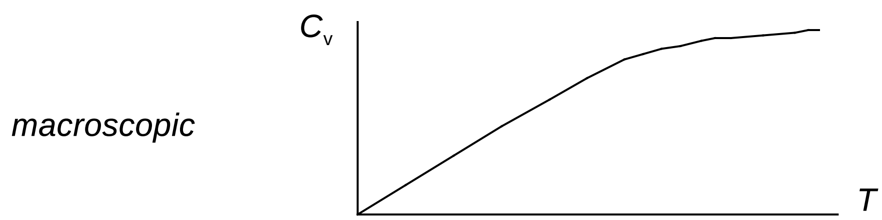
What's Important:
- galaxy formation
- galactic evolution

Text: Reif

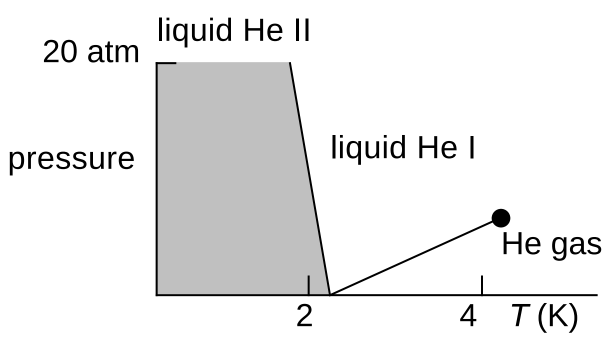
Phase transitions

Most of the systems considered so far in this course have consisted of non-interacting particles: we have calculated distribution functions, specific heats etc for MB, BE and FD statistics, and the special case of photons. In general, the behavior of all the quantities calculated has been smooth:





Up to now, the one exception has the condensation of ^{4}He atoms to form a superfluid



For ^{4}He, there is a phase transition - the discontinuous change in the amount of one component of a system and/or the discontinuous change in an observable such as the specific heat.

There are few systems which exhibit phase transitions without interactions between particles. Further, the dimensionality of a system may affect the presence of phase transitions. An argument advanced by Peierls illustrates why this should be so.

Ising model in 1D

Consider a system of spin - 1/2 particles equally spaced along a lattice, having only interactions with their nearest neighbours (Ising model). Because the particles are spin - 1/2, there are only two orientations of the spin vector. The interaction energy has a particularly simple form:

$E = 0$ for ↑↑

$E = J$ for ↑↓ (*i.e.* repulsive)

In one dimension, the spins all sit on a line

↑ ↓ ↓ ↑ ↓ ↑ ↑

The ground state of this system (lowest energy) has all spins aligned

$E = 0$ ↑ ↑ ↑ ↑ ↑ ↑ ↑

$\Omega(0) = 1$

only one state

$S = k_B \ln\Omega = 0$

The next highest state has one spin reversal along the line

$E = J$ ↑ ↑ ↑ ↓ ↓ ↓ ↓ ↓

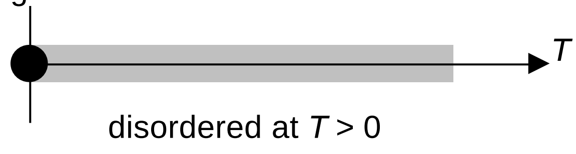
$\Omega(J) = N\text{-}1$

for N particles, there are N-1 positions to put the reversal

$S = k_B \ln\Omega = k_B \ln(N\text{-}1) \to \infty$ as $N \to \infty$

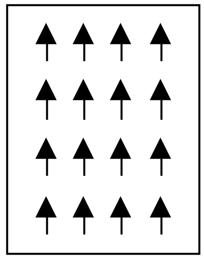
What does this imply about the free energy $\Delta F = \Delta E - T\,\Delta S$? The system is ordered at $T = 0$ (where entropy does not contribute) but disorders at any $T > 0$, because $-T\,\Delta S$ diverges logarithmically with N and dominates $\Delta E = J$ of the single spin reversal.

Thus, the phase diagram looks like

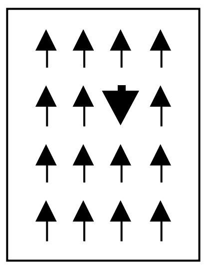


Ising model in 2D

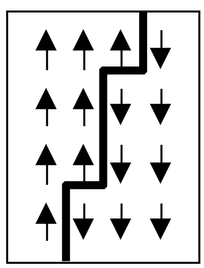
Now let's place the spin on a square lattice in two dimensions. In the ground state, the spins are all parallel and the energy is zero, just as with the one-dimensional case.



The first excited state involves the reversal of just one spin, and has an energy of $+4J$, the 4 arising from the number of nearest neighbours. This state does not represent a disordered system: it is only a local fluctuation involving $1/N$ of the particles of the system.

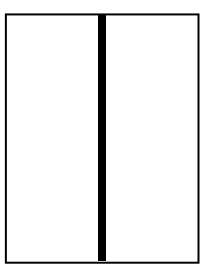


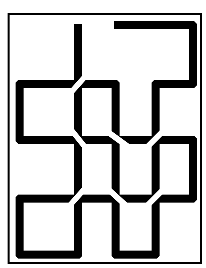
What destroys the overall order of the system is a state like



If the path length along the boundary between the two domains is L, then the energy required to create the boundary is JL. If the path runs all the way across the system, then L varies from $N^{1/2}$ to N^{1}. Some examples:

$L \sim N^{1/2}$





$L \sim N^{1}$

Clearly, there are more configurations for the situation on the right than on the left:

Left: $N^{1/2}$ starting points along each axis, since there are $N^{1/2}$ points in each direction. Thus, $\Omega = 2N^{1/2}$ for straight lines.

Right: Really convoluted paths would turn to the left or right at each lattice site; that is, there are 2 choices at each of L sites, or 2^L choices. The number of starting points along one axis is $N^{1/2}$, so the total number of configurations is of the order

$$\Omega \sim N^{1/2}\, 2^L$$

The 2^L factor overwhelms the $N^{1/2}$ term, leading to

$$\ln\Omega = (1/2)\ln N + L\ \ln 2.$$

Admittedly the argument for the RHS is a little crude, but taking the logarithm makes it less approximate.

Now, it costs energy to create the path on the right, and whether energy or entropy dominates the free energy depends on the temperature.

$$\Delta E \sim JL$$

$$T\,\Delta S = Tk_B\ln\Omega \sim Tk_B L\ \ln 2 = Lk_B T\ \ln 2.$$

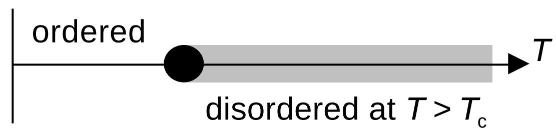
If $T\,\Delta S > \Delta E$, entropy dominates and the system is disordered. The temperature threshold for this is

$$Lk_B T_c\ \ln 2 = JL$$

or

$$T_c = J\,/\,(k_B \ln 2).$$

As a function of temperature, the phase diagram in two dimensions is



Summary

This simple argument predicts that

- 1D: system ordered at $T = 0$ only
- 2D: system ordered at $T < J\,/\,(k_B\ln 2)$ [*phase transition*]
- 3D: system ordered at $T < T_c$ [*phase transition, next lecture*]