Lecture 6 - Classical ideal gas

What's Important:

- classical phase space

Text: Reif

Classical ideal gas

In the previous lecture, we introduced the concept of accessible states, with the number of accessible states in a given energy range δE being defined as $\Omega(E)$, related to the density of states $\omega(E)$ via

$$\Omega(E) = \omega(E)\, \delta E. \qquad (6.1)$$

As a first example, we evaluate $\Omega(E)$ for a collection of classical non-interacting particles with only translational degrees of freedom. The energy of the system is

[*energy of system*] = [*kinetic energy of all particles*]
+ [*potential energy of all particles*]
+ [*interaction energy between particles*]
+ [*mass energy*]

or

$$E = K + U + E_{\text{int}} + Mc^2. \qquad (6.2)$$

For the classical ideal gas

$K = \Sigma_i p_i^2 / 2m_i$
[*potential energy of all particles*] = constant
[*interaction energy between particles*] = 0
[*mass energy*] = constant.

Let's make the further assumption that all particles have the same mass m, so

$$E = (1/2m)\, \Sigma_i p_i^2. \qquad (6.3)$$

The N particles in the system have $f = 3N$ degrees of freedom associated with their spatial coordinates $\mathbf{q}_1 \dots \mathbf{q}_N$. Thus, the number of states between E and $E + \delta E$ is proportional to the $6N$-dimensional volume of this phase space bounded by E and $E + \delta E$:

$$\Omega(E) \quad \iint \dots \iint d^3q_1 \dots d^3q_N\, d^3p_1 \dots d^3p_N. \qquad (6.4)$$

where the boundaries of the integrals are determined by E and $E + \delta E$.

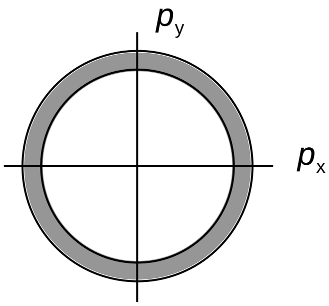
The generalized coordinates $\mathbf{q}_1 \dots \mathbf{q}_N$ can be chosen to be the Cartesian coordinates $\mathbf{r}_1 \dots \mathbf{r}_N$ here. Since the energy does not depend on the positions, then the integrals can be separated into spatial and momentum parts:

$$\Omega(E) \quad \iint d^3q_1 \dots d^3q_N \iint \dots d^3p_1 \dots d^3p_N. \qquad (6.5)$$

Since each particle in independent and can travel over the entire volume V, the set of coordinate-space integrals becomes $V_1 ... V_N = V^N$. Note that if the volume is doubled, but the energy is held fixed, then the number of accessible states increases by a factor of 2^N.

We are left with the task of evaluating $\iint .. d^3p_1 .. d^3p_N$ subject to the energy boundaries. To see how these integrals behave, we look at the specific case with $f = 2$. For a given value of E, the momentum vector lies between

$(2mE)^{1/2}$ and $[2m(E + \delta E)]^{1/2}$.



The "area" of the accessible momentum space is then proportional to

$2\pi p \, \delta p$,

where δp is set by δE.

We can generalize this to a sphere in f dimensions:

total number of states with energy less than E $p^f = E^{f/2}$

number of states with energy between E and $E + \delta E$ $E^{(f/2) - 1}$

Hence

$\Omega(E)$ $V^N E^{(f/2) - 1}$ where $f = 3N$.

What we see from this calculation is that the number of states Ω is proportional to V^N and $E^{(3N/2) - 1}$. In other words, Ω grows very fast with N, especially considering that $N \sim 10^{23}$ for everyday systems.

Reif presents a general argument based on single-particle states that $\Omega(E) \sim E^f$ (order of magnitude in the exponent).