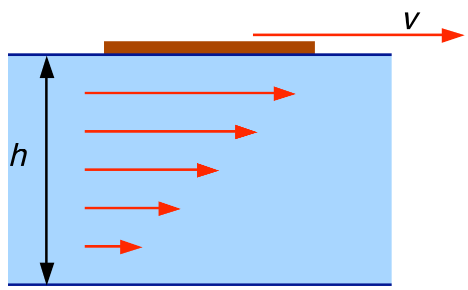
4xx Intro 5 - Forces and movement in a viscous environment

Even if the shear modulus vanishes, the response of a fluid to an applied force is not instantaneous. At low speeds, the response time of a fluid to accommodating an applied stress depends on the viscosity, η, among other factors. Consider one means of measuring η: flat plate of area *A* on one side is pulled along the surface of the fluid with a force *F*, giving a shear stress of *F/A*.



If the material in the figure were a solid, it would resist this stress until it attained a deformed configuration where the applied and reaction forces were in equilibrium. But a fluid doesn't resist shear, and the floating plate continues to move at a speed *v* as long as the stress is applied:

$$F/A = \eta\,(v/h), \tag{1}$$

where *h* is the height of the liquid in its container. Note that the fluid is locally stationary at its boundaries: it is at rest at the bottom of the container and moving with speed *v* beside the plate.

Elastic quantities such as the bulk modulus or shear modulus appear in Hooke's law expressions of the form [*stress*] = [*elastic modulus*]•[*strain*]. Strain is a dimensionless ratio like the change in volume divided by the undeformed volume, so elastic moduli must have the dimensions of stress. Eq. (1) is different from this, in that the ratio *v*/*h* is not dimensionless but has units of [*time*]$^{-1}$, so that η has dimensions of [*force*/*area*]•[*time*], or kg/m•s in the MKSA system. Thus, η provides the time scale for the relaxation, as expected. Viscosity is often quoted in units of Poise or P, which has the equivalence of kg/m•s ≡ 10 P.

Fluid	η (kg/m•sec)	η (P)
Air	1.8×10^{-5}	1.8×10^{-4}
Water	1.0×10^{-3}	1.0×10^{-2}
Olive oil	0.084	0.84
Glycerine	1.34	13.4
Glucose	10^{13}	10^{12}
mixtures: blood	2.7×10^{-3}	2.7×10^{-2}

Translational drag

The relationship between the drag force F_{drag} and the speed v depends on the shape of the object among other things, so for the time being we will simply write the relationship as

$$F_{drag} = c_1 v \qquad \text{(low speeds, streamline flow)} \qquad (2a)$$
$$F_{drag} = c_2 v^2, \qquad \text{(high speeds, turbulent flow)} \qquad (2b)$$

where the constants c_1 and c_2 depend on a variety of terms. Note that the power required to overcome the drag force, obtained from [*power*] = Fv, grows at least as fast as v^2 according to Eq. (2). Relatively speaking, viscous forces are so important in the cell that we need only be concerned with the low speed behavior of Eq. (2a).

Let's now solve the motion of an object subject only to linear drag in the horizontal direction, omitting gravity. The object obeys Newton's law F = ma = m (dv/dt), so that the drag force from Eq. (2a) gives the relation

$$ma = m\,(dv/dt) = -c_1 v, \qquad (3)$$

where the minus sign indicates that the force is in the opposite direction to the velocity. Eq. (3) can be rearranged to read

$$dv/dt = -(c_1/m)\ v, \qquad (4)$$

which relates a velocity to its rate of change. The solution is exponential in form, because

$$de^x/dx = e^x. \qquad (5)$$

One can verify Eq. (5) by explicit substitution, finding

$$v(t) = v_o \exp(-c_1 t\ /\ m), \qquad (6)$$

where v_o is the speed of the object at $t = 0$.

The characteristic time scale for the velocity to decay to $1/e$ of its original value is m/c_1. The time-dependence of the distance can be found by integrating Eq. (2.15) to yield:

$$\Delta x = (mv_o\ /\ c_1) \bullet [1 - \exp(-c_1 t\ /\ m)], \qquad (7)$$

where the limiting value at $t \rightarrow \infty$ is $x = mv_o\ /\ c_1$.

The drag force also depends on the cross sectional shape that is presented to the fluid by the object in its direction of motion.

Stokes' law for a sphere of radius R

$$F = 6\pi\eta Rv. \tag{8}$$

An ellipsoid of revolution with semi-major axis *a* and semi-minor axis *b*

$$F = 4\pi\eta av \,/\, \{\ln(2a/b) - 1/2\}, \tag{9}$$

when $a >> b$ for motion at low speed parallel to the long axis of the ellipsoid.

At higher speeds, the drag force for translational motion depends on the square of the speed and the shape of the object:

$$F = (\rho/2)AC_D v^2, \tag{10}$$

where ρ is the density of the fluid and A is the cross sectional area of the object in its direction of motion (πR^2 for a sphere). The dimensionless drag coefficient C_D is often about 0.5 for many shapes of interest. Note that the drag force in Eq. (10) depends on the density of the fluid, rather than its viscosity η in Eqs. (8) and (9).

Example: Consider an idealized bacterium swimming in water, assuming:
- the bacterium is a sphere of radius $R = 1$ μm
- the fluid medium is water with $\eta = 10^{-3}$ kg / m•s
- the density of the cell is that of water, $\rho = 1.0 \times 10^3$ kg/m^3
- the speed of the bacterium is $v = 2 \times 10^{-5}$ m/s.

What is the drag force experienced by the cell? If the cell's propulsion system were turned off, over what distance would it come to a stop (ignoring thermal contributions to the cell's kinetic energy from the its environment)?

First, we calculate the prefactor c_1 in Eq. (2a)

$$c_1 = 6\pi\eta R = 6\pi \cdot 10^{-3} \cdot 1 \times 10^{-6} = 1.9 \times 10^{-8} \text{ kg/s},$$

so that the drag force on the cell can then be obtained from Stoke's law:

$$F_{drag} = c_1 v = 1.9 \times 10^{-8} \cdot 2 \times 10^{-5} = 0.4 \text{ pN}. \qquad (\text{pN} = 10^{-12} \text{ N})$$

To determine the maximum distance that the cell can drift without propulsion, we first calculate the mass of the cell m,

$$m = \rho \cdot 4\pi R^3 /3 = 10^3 \cdot 4\pi\,(1 \times 10^{-6})^3 /3 = 4.2 \times 10^{-15} \text{ kg},$$

from which the stopping distance becomes, using Eq. (7)

$$x = mv_o /c_1 = 4.2 \times 10^{-15} \cdot 2 \times 10^{-5} / 1.9 \times 10^{-8} = 4.4 \times 10^{-12} \text{ m} = 0.04 \text{ Å}.$$

Rotational drag

The stress experienced by the surface of an object moving through a viscous fluid can retard the rotational motion of the object, as well as its translational motion. The effect of rotational drag is to produce a torque τ that reduces the object's angular speed ω with respect to the fluid. At low angular speed, the torque from drag is linearly proportional to ω, just as the linear relation Eq. (2a) governs translational drag:

$$\tau = -\chi\omega. \tag{11}$$

where the minus sign indicates τ acts to reduce the angular speed (counter-clockwise rotation corresponds to positive ω). For a sphere of radius R, the drag parameter χ is

$$\chi = 8\pi\eta R^3, \tag{12}$$

and for an ellipsoid of revolution

$$\chi = (16/3)\,\pi\eta a b^2 \qquad (a >> b) \tag{13}$$

where η is the viscosity of the medium.

It's straightforward to solve for the functional form $\omega(t)$ of the angular speed and $\theta(t)$ of the angle swept out by the object. For instance, if the rotation is about the longest or shortest symmetry axis of the object, then the torque produces an angular acceleration α that determines $\omega(t)$ *via*

$$\tau = I\alpha = I\,(d\omega/dt) = -\chi\omega, \tag{14}$$

where I is the moment of inertia about the axis of rotation. For a sphere of radius R, $I = mR^2/2$. Eq. (14) determines the functional form of $\omega(t)$:

$$\omega(t) = \omega_o \exp(-\chi t\,/\,I), \tag{15}$$

where ω_o is the initial value of ω. Integrate Eq. (15) to obtain $\theta(t)$.

Example: Consider an idealized bacterium swimming in water, assuming:

- the bacterium is a sphere of radius $R = 1$ μm
- the fluid medium is water with $\eta = 10^{-3}$ kg / m•s
- the bacterium rotates at a frequency of 10 revolutions per second.

Find the retarding torque from drag experienced by the cell.

First, the frequency of 10 revolutions per second corresponds to an angular frequency of $\omega = 20\pi$ s^{-1}. Next, the prefactor χ in Eq. (12) is

$$\chi = 8\pi\eta R^3 = 8\pi \cdot 10^{-3} \cdot (1 \times 10^{-6})^3 = 8\pi \times 10^{-21} \text{ kg-m}^2/\text{s},$$

so that the magnitude of the drag torque on the cell can then be obtained from:

$$\tau_{drag} = \chi\omega = 8\pi \times 10^{-21} \cdot 20\pi = 1.6 \times 10^{-18} \text{ N-m}.$$

As a final caveat, most readers with a physics background are aware that the kinematic quantities ω, α, and τ are vectors and I is a tensor. Thus, the situations we have described are specific to rotations about a particular set of axes through an object.

When ω and τ have arbitrary orientations with respect to the symmetry axes, the motion is more complex than what has been described here.

Reynolds number

As seen in the examples, the effect of drag easily overwhelms the cell's inertial movement at constant velocity that follows Newton's First Law of mechanics. In fluid dynamics, a benchmark exists for estimating the importance of the inertial force compared to the drag force. This is Reynolds number, a dimensionless quantity given by

$$\mathbf{R} = \rho\, v\, \ell \,/\, \eta, \tag{16}$$

where v and ℓ are the speed and length of the object, and ρ and η are the density and viscosity of the medium, all respectively. The crossover between drag-dominated motion at small $\mathbf{R}$ and inertia dominated motion at large $\mathbf{R}$ is in the range $\mathbf{R} \sim$ 10-100.

Let's collect the terms on the right hand side of Eq. (16) into properties of the fluid (ρ/η) and those of the object ($v\ell$); for water at room temperature, ρ/η is 10^6 s/m^2. Common objects like fish and boats, with lengths and speeds of metres and metres per second, respectively, have $v\ell$ in the range of 1-1000 m^2/s. Thus, $\mathbf{R}$ for everyday objects moving in water is 10^6 or more, and such motion is dominated by inertia, even though viscous effects are present. This conclusion also applies for cars and planes as they travel through air, where ρ/η is 0.5×10^5 s/m^2 under standard conditions. However, for the motion of a cell, the product $v\ell$ is far smaller: even if $\ell = 4$ μm and $v = 20$ μm/s, then $v\ell = 8 \times 10^{-11}$ m^2/s, such that $\mathbf{R}$ is less than 10^{-4}. Clearly, this value is well below unity so the motion of a typical cell is dominated by viscous drag. In the context of Reynolds number, the reason for this is the very small size and speed of cells compared to everyday objects.