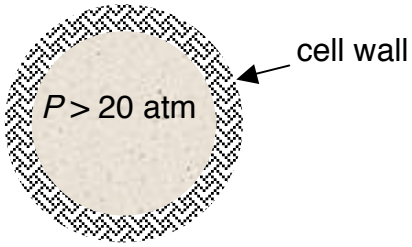
PHYS 4xx Net 6 - Network percolation and failure

Strain at failure

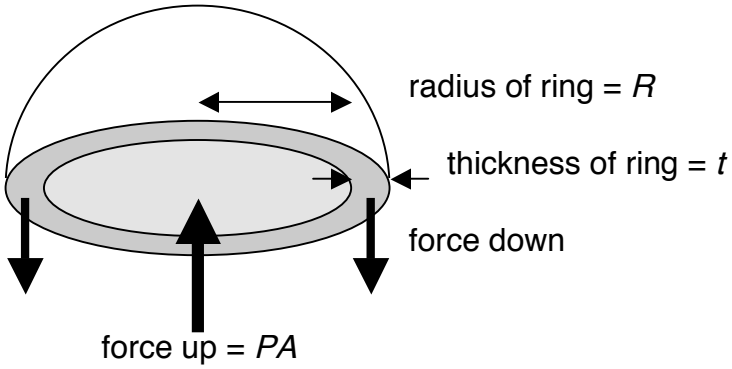
- strain at which a material fails varies widely
- many lipid bilayers fail at 2-5% strain
- some protein networks can withstand strains of 30% or more
- brittle materials fail at very small strains

Stress at failure

Let's use a pressurized bacterium to provide a benchmark for the stresses some components of the cell must withstand.



Take a section through the cell, and applying a force balance equation at the cell wall:



Now, the force acting up across the area of the section is

F_{UP} = [*pressure*]•[*area*]

or

$F_{UP} = \pi R^2 P$

where R is the radius of the cell. The force is **not** $2\pi R^2 P$.

The force acting down from the cell wall (which has the shape of a ring in cross section) has a similar form, with the 3D stress replacing the pressure in the fluid.

F_{DOWN} = [3D *stress*] • [*area of the ring*]

If the thickness of the shell is small, $t << R$, then the area of the ring is

[*area*] = [*perimeter*] • [*thickness*] = $2\pi Rt$.

Equating the forces gives

$F_{DOWN} = F_{UP}$

[3D *stress*] • $2\pi Rt = P\,\pi R^2$

or

[3D *stress*] $t = PR / 2$.

This is known as the law of Laplace, which was originally derived for soap bubbles.

OK, so let's put in some numbers.

radius of bacterium = 1 μm

[3D *stress at failure*] = 2 x 10^7 J/m^3 (typical value for many materials)

maximum pressure required = 20 atm = 2 x 10^6 J/m^3.

Then the predicted thickness of the cell wall to carry this stress is

$t = PR / (2[stress])$

$= 2 \times 10^6 \times 10^{-6} / (2 \times 2 \times 10^7)$

= 50 nm

This is pretty close for Gram positive bacteria that have to withstand these pressures. Antibiotics take advantage of these high pressures to destroy bacteria.

Griffith formula

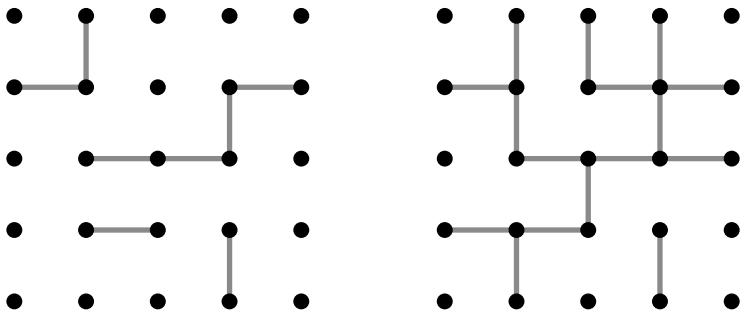
The failure of brittle materials has been well studied; one of the fundamental results is that the critical stress σ_c for the propagation of a crack is given by

$$\sigma_c \sim (2Y\gamma_s / \pi a)^{1/2},$$

where Y is the Young's modulus, γ_s is the interfacial tension between the exposed surface of the crack and the medium. The crack length is $2a$.

Percolation

What happens if the chain density is so low that a connected lattice doesn't exist? Consider the two-dimensional square lattice, on which bonds have been placed randomly



p = probability that a given bond site is occupied (p = 1 for no vacancies)
p is low on the left hand configuration, and no connected path crosses the lattice
p is higher on the right hand side (but still < 1) and a connected path exists
for infinite systems, a connecting path appears at the connectivity percolation threshold p_C

p_C = 0.5 for a square lattice in 2D and $p_C \sim 0.35$ for a triangular lattice in 2D

Feng and Sen (1984) showed that the elastic moduli vanish below a distinct rigidity percolation threshold p_R, which may be larger than p_C at zero temperature

One argument based on the counting of contraints gives (Maxwell, 1864)
number of bonds linked to a given vertex is, on average
[*no. of bonds connected per site*] = zp
where z is the number of links available

for N sites, the total number of bonds is
[*total no. bonds*] = $Nzp/2$
where the factor of 2 comes from each bond being shared by two sites

Now, there are Nd degrees of freedom for the lattice in a d-dimensional space, so the net number of degrees of freedom after applying the bond constraints is
[*no. of floppy modes*] = $Nd - Nzp/2$.

at the failure point, the number of floppy modes vanishes, so the occupation probability at failure p^* is
$p^* = 2d/z$

Measured value of μ for red blood cells decreases with spectrin density; appears to be closer to p_C than p_R

Percolation threshold for a selection of regular lattices in two and three dimensions

Lattice	z	p_C (bond)	p_C (site)	p^* (bond)
Two dimensions				
honeycomb	3	0.653	0.696	1
square	4	0.500	0.593	1
triangular	6	0.347	0.500	2/3
Three dimensions				
simple cubic	6	0.249	0.312	1
body-centered cubic	8	0.180	0.246	3/4
face-centered cubic	12	0.119	0.198	1/2

"Bond" and "site" refer to different measures of defects in the lattice, as described in the text; Data from Chapter 2 of Stauffer and Aharony (1992).