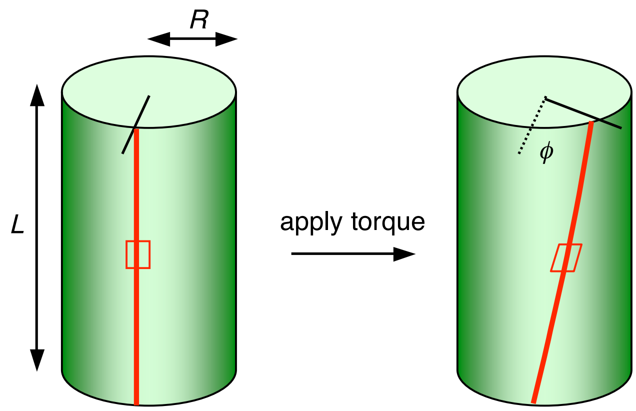
PHYS 4xx Poly 5 - Torsion, twist and writhe

Description of the twist deformation

Fix one end of a uniform cylinder and apply a torque around the cylindrical axis to the other end. The rotational angle ϕ is proportional to the torque $\mathcal{T}$

$$\mathcal{T} \propto \phi. \qquad (1)$$



The *twist* of the cylinder α, is the rate of change of the rotational angle ϕ as a function of the length of the cylinder, or

$$\alpha \equiv \phi \,/\, L. \qquad (2)$$

twist, no curvature

Twist is not an angle, it has units of inverse length and is constant along the cylinder. It is positive or negative according to the sign of ϕ. Writing Eq. (1) in terms of twist

$$\mathcal{T} = \kappa_{tor}\alpha, \qquad (3)$$

where κ_{tor} is the torsional rigidity, the analog of the flexural rigidity κ_f for bending. Note that $\mathcal{T} = (\kappa_{tor}/L)\ \phi$, is the analog of the spring equation $F = (YA/L)\ x$

Considering the small rectangle drawn on the side of the cylinder, twisting the rod corresponds to shearing the rectangle, and we expect κ_{tor} to be proportional to the shear modulus μ. For a uniform cylinder of radius R,

$$\kappa_{tor} = \mu\,\pi R^4/2 \qquad (4)$$

solid cylinder

which has a very similar form as the flexural rigidity $\kappa_f = Y\,\pi R^4/4$; both κ_{tor} and κ_f have units of [*energy*]•[*length*].

From Eq. (4), the torsional rigidity of a hollow tube must have the form

$$\kappa_{tor} = \mu\,\pi(R_{outer}{}^4 - R_{inner}{}^4)/2. \qquad (5)$$

hollow cylinder

Further, a solid rod with the cross sectional shape of an ellipse obeys

$$\kappa_{tor} = \mu\,\pi a^3 b^3 \,/\, (a^2 + b^2), \qquad (6)$$

elliptical rod

where a and b are the semi-major and semi-minor axes of the ellipse.

The energy density per unit length $\mathcal{E}$ of the twist deformation can be obtained by integrating Eq. (3) over α

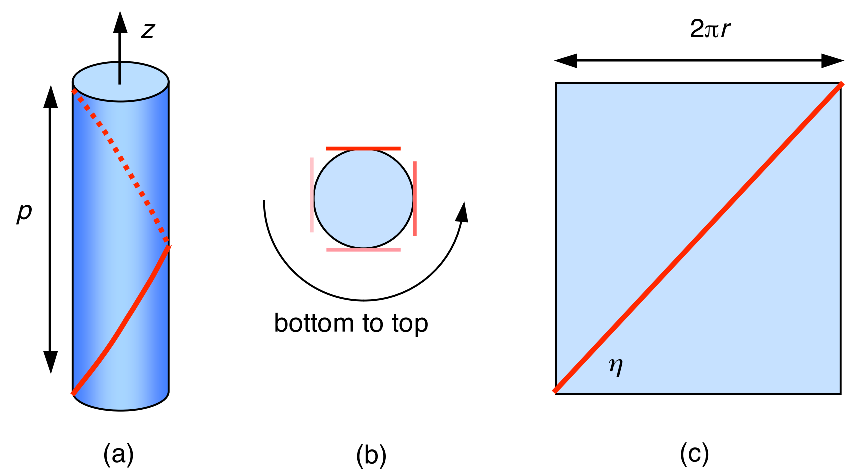
$$\mathcal{E} = \kappa_{tor}\alpha^2/2. \qquad (7)$$

The Young's modulus of many solids is about two or three times the shear modulus, so the torsional rigidity ($\mu \pi R^4/2$) is similar in magnitude to the flexural rigidity ($Y \pi R^4/4$). Indeed if $Y = 2\mu$, then $\kappa_{tor} = \kappa_f$. Some measured values are:

Filament	κ_{tor} (J•m)	κ_{tor}/k_BT (m)	κ_f/k_BT (m)
DNA	$(2 - 4) \times 10^{-28}$	$(50 - 100) \times 10^{-9}$	53×10^{-9}
F-actin	$(3 - 8) \times 10^{-26}$	$(7.5 - 20) \times 10^{-6}$	$(10 - 20) \times 10^{-6}$

Twist with curvature

Because $\kappa_{tor} \sim \kappa_f$, the deformation of a beam under torsion may involve twisting of the beam along its symmetry axis as well as twisting of the axis itself into a helical shape:



The red line on the cylinder is the axis of a rectangular ribbon, whose shape is shown in cross section in the middle panel.

The *pitch* (p) of the beam is the length along the helical axis during which the beam completes one rotation as a helix; in the diagram, the plane of the ribbon also completes one rotation in this interval in this situation. If the beam were straight, then $p = 2\pi/\alpha$ from Eq. (2) when $\phi = 2\pi$ and $L = p$, but this relationship is not true in general.

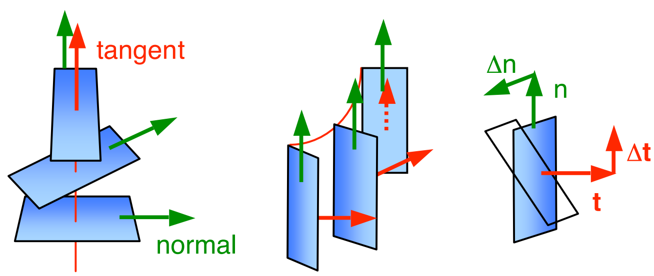
"Unroll" the surface of the imaginary cylinder without changing the location of the beam on it. The "height" of the beam's path (its projection on the z-axis) increases linearly with the rotational angle around the axis because α is constant; consequently the beam executes a diagonal on the unrolled surface in panel (c). The base of this rectangular surface is $2\pi r$, and the overall height is just the pitch p, so the length s_{helix} of the beam in one circuit around the helix must be

$$s_{\text{helix}} = [(2\pi r)^2 + p^2]^{1/2}. \quad (8)$$

From panel (c), the angle η between the beam and the plane perpendicular to the helical axis is

$$\tan\eta = p / 2\pi r. \quad (9)$$

The beam twists around its axis at the same time as that axis follows a helical path through space.



(d) (e) (f)

Panel (d): a series of rectangles as their orientation twists around a straight-line path. The unit tangent vector **t** to the *path* remains fixed, so $\Delta\mathbf{t} = 0$. But the unit vector **n** normal to the path and attached to the rectangle rotates around the axis of the path, so $\Delta\mathbf{n} \neq 0$. In panel (d), both **n** and Δ**n** are perpendicular to **t**.

Panel (e): the normal vectors to the path do not change orientation, so $\Delta\mathbf{n} = 0$, while the tangent vectors to the path rotate along the path with $\Delta\mathbf{t} \neq 0$. In panel (e), both **t** and Δ**t** are perpendicular to **n**.

Panel (f), a change in the orientation of the cross section through a beam involves Δ**t** in and Δ**n**. As drawn, these changes are orthogonal, so the magnitude of the total change is $(\Delta\mathbf{t}^2 + \Delta\mathbf{n}^2)^{1/2}$.

For small arcs Δs along the path of the helix:
$|\Delta\mathbf{t}| = C \bullet \Delta s$ (recall θ subtended by Δs is $\Delta s/R_c = C \bullet \Delta s$; θ is also $|\Delta\mathbf{t}|$)
$|\Delta\mathbf{n}| = \alpha \bullet \Delta s$ from the definition of α as $\Delta\phi / \Delta s$
Since the total change in orientation is just 2π over the arc length s_{helix}, then the change that occurs over Δs must be $2\pi\,(\Delta s / s_{\text{helix}})$. Hence,

$$[2\pi\,(\Delta s / s_{\text{helix}})]^2 = (C\,\Delta s)^2 + (\alpha\,\Delta s)^2, \quad (10)$$

or

$$(2\pi / s_{\text{helix}})^2 = C^2 + \alpha^2. \quad (11)$$

Use Eq. (8) to eliminate s_{helix} from Eq. (11), leaving

$$(2\pi)^2 / (C^2 + \alpha^2) = (2\pi r)^2 + p^2. \quad (12)$$

A second relationship for α and C comes from evaluating $|\Delta\mathbf{n}| / |\Delta\mathbf{t}|$:

- From the steps leading to Eq. (10), $|\Delta\mathbf{n}| / |\Delta\mathbf{t}| = \alpha/C$.
- But $|\Delta\mathbf{n}| / |\Delta\mathbf{t}| = \tan\eta$. The proof is:

At an angle , **t** rotates around the helical axis, covering a distance of $2\pi \cos\eta$ in one complete revolution. Thus, for a given Δs, $|\Delta\mathbf{t}| = 2\pi \cos\eta \, (\Delta s / s_{helix})$.

Similarly, **n** rotates around the helical axis, but at a different radius, covering a distance of $2\pi \sin\eta$ in one complete revolution. Thus, $|\Delta\mathbf{n}| = 2\pi \sin\eta \, (\Delta s / s_{helix})$.

Combining these equations yelds $|\Delta\mathbf{n}| / |\Delta\mathbf{t}| = \tan\eta$.

But $\tan\eta = p / 2\pi r$ in panel (c). Thus,

$$\alpha/C = |\Delta\mathbf{n}| / |\Delta\mathbf{t}| = p / 2\pi r \qquad (13)$$

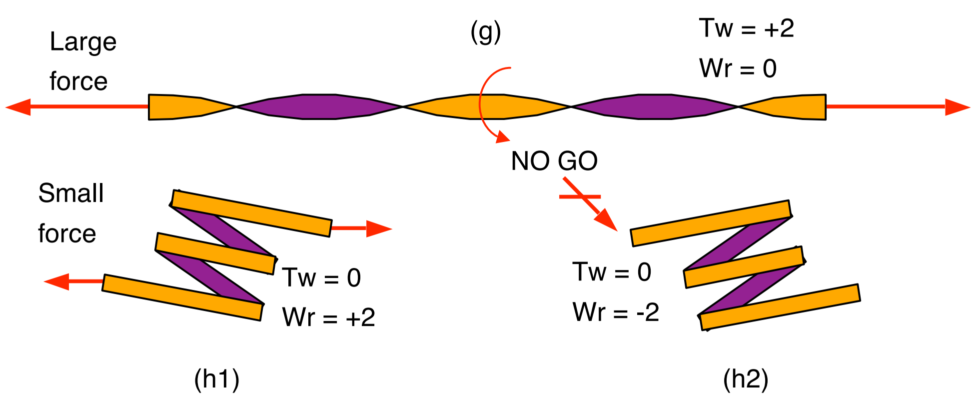
Use Eq. (13) to express either p or r in terms of α and C by substitution into Eq. (12):

$$r = C / (C^2 + \alpha^2) \qquad p = 2\pi\alpha / (C^2 + \alpha^2) \qquad (14)$$

and

$$\alpha = 2\pi p / (4\pi^2 r^2 + p^2) \qquad C = 4\pi^2 r / (4\pi^2 r^2 + p^2) \qquad (15)$$

Twist and writhe



Because $\kappa_{tor} \sim \kappa_f$, when a torque is applied to a beam, it may be more favourable for the beam to deform into a helix than to retain its axis of cylindrical symmetry; which configuration is more favoured depends on the cross sectional shape of the beam. For example, a belt can be twisted fairly easily into the form displayed in panel (g), where one end of the belt has been twisted through two complete rotations about its long axis. Large enough longitudinal forces are applied to the opposite ends of the belt to keep its axis straight. If the forces are reduced, we know from experience that the belt may untwist slowly as two loops appear, shown in panel (h1). Note that configurations (g) cannot deform into configuration (h2) if the orientation of the ends is kept fixed.

Twist (*Tw*) and writhe (*Wr*) describe the overall topology of rods. Twist is the number of complete turns made by a vector normal to the axis of the rod (and in its plane) as it is propagated from one end to the other.

Panel (g): the plane of the rod rotates about the axis twice from left to right, in a right-handed spiral. Thus, $Tw = +2$, where the plus sign indicates right-handedness.

Panels (h1) and (h2): a normal to rod's axis (and lying in the plane of the rod) does not change direction around the loop, and so both of these configurations have $Tw = 0$.

In the context of a beam or ribbon with free ends (*i.e.* ends that are not attached to each other), the writhe is the number of loops, taking into account the handedness of the spiral. In panel (h1), the writhe is +2 because there are two complete loops and the spiral is right-handed, while the mirror image of this configuration is left-handed with $Wr = -2$, as shown in panel (h2).

The algebraic sum of *Tw* and *Wr* must be constant in a continuous deformation. The sum is called the linking number, *Lk*:

$$Lk = Tw + Wr, \tag{16}$$

where the signs of *Tw* and *Wr* must be taken into account. For example, configurations (g) and (h1) both have $Lk = +2$, and are continuously deformable into each other, while configuration (h2) has $Lk = -2$ and consequently is not accessible from the other configurations. The fact that *Lk* does not change under a continuous deformation ($\Delta Lk = 0$) means that ΔTw and ΔWr are correlated: $\Delta Tw = -\Delta Wr$ according to Eq. (16).