State Preparation, Determination, Measurement

Chaps. 8 and 9 of Ballentine deal with state preparation and measurement. A number of the ideas and topics of these chapters have been incorporated into previous sections of these notes. All that we wish to cover here are the indeterminacy relations.

When we think of measuring an observable as in $Tr(\hat{\rho}\hat{R}) = \langle R \rangle$, we use statistical ideas and talk about the expectation of an observable and, naturally, its variance $\Delta_A{}^2 = \langle A \rangle^2 - \langle A^2 \rangle$. However, in many situations $\Delta_B{}^2$ depends on the expectations of other observables in the system. Let's consider the following situation:

We have two dynamical observables $\mathcal{A}$ and $\mathcal{B}$ with corresponding operators $\hat{A}$ and $\hat{B}$. Suppose $\hat{A}$ and $\hat{B}$ don't commute:

$$[\hat{A}, \hat{B}] = i\hat{C} \qquad \left(\begin{array}{c}\text{the factor of } i \text{ ensures} \\ \hat{C}^{+} = \hat{C}\end{array}\right).$$

Let's define two operators $\hat{A}_0 = \hat{A} - \langle \mathcal{A} \rangle$, $\hat{B}_0 = \hat{B} - \langle \mathcal{B} \rangle$, so that

$$\Delta_A{}^2 = Tr(\hat{A}_0^2 \hat{\rho}) \qquad \Delta_B{}^2 = Tr(\hat{\rho}\hat{B}_0^2)$$

For any operator T, we have $Tr(\hat{\rho}\hat{T}\hat{T}^{+}) \geq 0$

Proof.
Use $Tr(\hat{\rho}\hat{T}\hat{T}^{+}) = Tr(\hat{T}^{+}\hat{\rho}\hat{T}) = Tr([UT^{+}U^{-1}][U\rho U^{-1}][UTU^{-1}])$

choose U to diagonalize this

$$= Tr(\tilde{T}^{+}\rho_D \tilde{T})$$

$$= \sum_n \langle n | \tilde{T}^{+}\tilde{T} | n \rangle \geq 0 \qquad \leftarrow Tr\rho_D = \sum_n |n\rangle\langle n|$$

Now, let's substitute

$$\hat{T} = \hat{A}_o + i\omega\hat{B}_o \quad \text{and} \quad \hat{T}^+ = \hat{A}_o - i\omega\hat{B}_o$$

$$\Rightarrow \mathrm{Tr}(\hat{\rho}\hat{T}\hat{T}^+) = \mathrm{Tr}(\hat{\rho}\hat{A}_o^2) - i\omega\,\mathrm{Tr}(\hat{\rho}[\hat{A}_o, \hat{B}_o]) + \omega^2\,\mathrm{Tr}(\hat{\rho}\hat{B}_o^2) \geq 0$$

But $\underbrace{[\hat{A}, \hat{B}] = [\hat{A}_o, \hat{B}_o]}_{\text{just adding constants to each expression, which commute.}} = i\hat{C}$

$$\Rightarrow \mathrm{Tr}(\hat{\rho}\hat{A}_o^2) + \omega\,\mathrm{Tr}(\hat{\rho}\hat{C}) + \omega^2\,\mathrm{Tr}(\hat{\rho}\hat{B}_o^2) \geq 0 .$$

This is a quadratic in ω. Let's put the r.h.s. $= K$, and find the extremum for ω:

$$\omega = \frac{-\mathrm{Tr}(\hat{\rho}\hat{C}) \pm [\,\mathrm{Tr}(\hat{\rho}\hat{C})^2 - 4\,\mathrm{Tr}(\hat{\rho}\hat{B}_o^2)(\mathrm{Tr}(\hat{\rho}\hat{A}_o^2) - K)\,]^{1/2}}{2\,\mathrm{Tr}(\hat{\rho}\hat{B}_o^2)} .$$

The largest value of K corresponds to the smallest value of the $[\quad]^{1/2}$. The corresponding value of ω is ω_{max}:

$$\omega_{max} = \frac{-\mathrm{Tr}(\hat{\rho}\hat{C})}{2\,\mathrm{Tr}(\hat{\rho}\hat{B}_o^2)}$$

and the corresponding value for $\mathrm{Tr}(\rho T T^+)$ is

$$\mathrm{Tr}(\hat{\rho}\hat{A}_o^2) \quad -\frac{[\mathrm{Tr}(\hat{\rho}\hat{C})]^2}{2\,\mathrm{Tr}(\hat{\rho}\hat{B}_o^2)} + \frac{[\mathrm{Tr}(\hat{\rho}\hat{C})]^2}{[2\,\mathrm{Tr}(\hat{\rho}\hat{B}_o^2)]^2} \cdot \mathrm{Tr}(\hat{\rho}\hat{B}_o^2)$$

$$= \mathrm{Tr}(\hat{\rho}\hat{A}_o^2) - \frac{[\mathrm{Tr}(\hat{\rho}\hat{C})]^2}{4\,\mathrm{Tr}(\hat{\rho}\hat{B}_o^2)} \geq K$$

$$\Rightarrow \mathrm{Tr}(\hat{\rho}\hat{A}_o^2) \cdot \mathrm{Tr}(\hat{\rho}\hat{B}_o^2) - \frac{[\mathrm{Tr}(\rho\hat{C})]^2}{4} \geq K \cdot \mathrm{Tr}(\hat{\rho}\hat{B}_o^2) \geq 0 .$$

$$\Rightarrow \Delta_A^2 \cdot \Delta_B^2 - \frac{\langle C \rangle^2}{4} \geq 0 .$$

or finally: $\Delta_A \cdot \Delta_B \geq \frac{1}{2}|\langle C \rangle|$

Two examples:

__Angular momentum__: $[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z$.

$$\Rightarrow \quad \Delta^2_{J_x} \cdot \Delta^2_{J_y} \geq \left(\frac{\hbar}{2}\right)^2 \langle J_z \rangle^2$$

If the expectation is over $\langle j, m|$, then

$$\Delta^2_{J_x} \cdot \Delta^2_{J_y} \geq \left(\frac{m\hbar}{2}\right)^2 .$$

Further, if the system is rotationally invariant

$$\langle \hat{J}_x \rangle = 0 \; ; \; \langle \hat{J}_y \rangle = 0 \quad \text{and} \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle$$

$$\therefore \; \Delta^2_{J_x} = \langle \hat{J}_x^2 \rangle \quad ; \quad \Delta^2_{J_y} = \langle \hat{J}_y^2 \rangle .$$

But $\langle J^2 \rangle \left(= j(j+1)\hbar^2\right) = \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle$

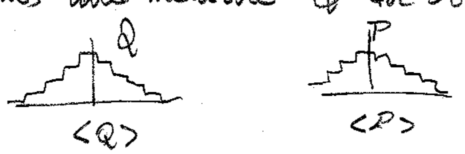
$$= 2\langle J_x^2 \rangle + m^2\hbar^2$$

$$\therefore \; \langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{\hbar^2}{2}\left(j(j+1) - m^2\right) \quad (\text{or } 2\langle J_y^2 \rangle) .$$

__Position & Momentum__ $[\hat{Q}, \hat{P}] = i\hbar$

$$\Rightarrow \quad \Delta_Q \cdot \Delta_P \geq \tfrac{1}{2}\hbar .$$

Note: this relationship does not say the Q, P must be measured simultaneously. It just says that you take an ensemble of systems and measure $\hat{Q}$ for some, $\hat{P}$ for others, then you find distributions



with $\Delta_Q \cdot \Delta_P \geq \frac{\hbar}{2}$.