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Physical consequences of a solution of the nonsymmetric unified field theory*

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The physical consequences and possible tests of our solution of the nonsymmetric unified field theory of gravitation and electromagnetism are discussed. It is found in general that a universal constant k introduced in the unification of the theory shows itself as a fourth-order effect in the propagation of light near a charged body. Further, it is shown that magnetic monopoles are not an allowed solution of the theory.

I. INTRODUCTION

The unification of Einstein's theory of general relativity and Maxwell's theory of electromagnetism has been an outstanding problem in physics for some time. Previous attempts at unification either resulted in the incorrect equations of motion of charged particles, or led to self-contradictory solutions. Recently,[1] we found that a modification of Einstein's nonsymmetric unified field theory led to a solvable and consistent set of equations.

Starting from the real nonsymmetric tensor $g_{\mu\nu}$ one can obtain the affine connection and the contracted curvature tensor from the definitions[2]

$$\partial_\alpha g_{\mu\nu} - g_{\mu\sigma}\Gamma^\sigma_{\alpha\nu} - g_{\sigma\nu}\Gamma^\sigma_{\mu\alpha} = 0 \tag{1.1}$$

and

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \tfrac{1}{2}(\partial_\nu \Gamma^\alpha_{(\mu\alpha)} + \partial_\mu \Gamma^\alpha_{(\nu\alpha)}) - \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\alpha\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\alpha\sigma} , \tag{1.2}$$

where

$$g_{(\mu\nu)} = \tfrac{1}{2}(g_{\mu\nu} + g_{\nu\mu}) , \quad g_{[\mu\nu]} = \tfrac{1}{2}(g_{\mu\nu} - g_{\nu\mu}) . \tag{1.3}$$

The contravariant tensor $g^{\mu\nu}$ is obtained by the relation

$$g^{\mu\alpha} g_{\nu\alpha} = \delta^\mu_\nu . \tag{1.4}$$

Our modified field equations are then

$$\Gamma^\alpha_{[\mu\alpha]} = 0 , \tag{1.5}$$

$$R^*_{(\mu\nu)} = 0 , \tag{1.6}$$

$$_{[\sigma} R^*_{[\mu\nu]]} = 0 . \tag{1.7}$$

The tensor $R^*_{\mu\nu}$ is defined by

$$R^*_{\mu\nu} = R_{\mu\nu} + I_{\mu\nu} , \tag{1.8}$$

where

$$I_{\mu\nu} = -\frac{1}{2k^2}(g_{\mu\sigma} g^{[\sigma\rho]} g_{\rho\nu} + \tfrac{1}{2} g_{\mu\nu}\, g_{\sigma\rho} g^{[\sigma\rho]} + g_{[\mu\nu]}) \tag{1.9}$$

and k is a universal constant.

It can be shown that Eq. (1.5) is equivalent to

$$\partial_\alpha \mathfrak{g}^{[\mu\alpha]} = 0 , \tag{1.10}$$

where

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} \tag{1.11}$$

and g is the determinant of $g^{\mu\nu}$.

If the antisymmetric part of $g_{\mu\nu}$ vanishes, then so does $I_{\mu\nu}$ and the field equations simply reduce to those of general relativity. We also define a tensor $g'_{[\mu\nu]}$ by

$$g_{[\mu\nu]} = k g'_{[\mu\nu]} \tag{1.12}$$

and make the identification

$$g'_{[\mu\nu]} = F_{\mu\nu} , \tag{1.13}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. In the limit that $k \to 0$, $R_{\mu\nu}$ "decouples" from the antisymmetric tensor $g'_{[\mu\nu]}$, and we find that Eqs. (1.7) and (1.10) become Maxwell's covariant equations of electromagnetism

$$F^{\mu\nu}{}_{;\nu} = 0 , \tag{1.14}$$

$$_{[\sigma} F_{\mu\nu]} = \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 . \tag{1.15}$$

Here, the subscripted semicolon denotes absolute differentiation with respect to the Christoffel symbols $\{{}_\mu{}^\lambda{}_\nu\}$. In this same limit, Eq. (1.6) becomes

$$G_{\mu\nu} = -8\pi T_{\mu\nu} , \tag{1.16}$$

where $G_{\mu\nu}$ is the contracted Riemann-Christoffel tensor and $T_{\mu\nu}$ is the energy-density tensor of the electromagnetic field. This could be regarded as a correspondence principle, such that our theory reduces to the Einstein-Maxwell theory of gravitation and electromagnetism as a special case, when the universal constant $k \to 0$. The obvious question which we must now answer is: What is the value of the universal constant k?

II. ELECTRIC FIELD SOLUTIONS

To answer the above question, we must find out how k appears in the metric generated by our field equations. Let us first look at the simplest example, that of a spherically symmetric electric field. In this case, the only nonzero element of $g_{[\mu\nu]}$ is $g_{[14]}$, which we identify with the electric field E by

$$g_{[14]} = kE \ . \tag{2.1}$$

The solution of the tensor $g_{\mu\nu}$ leads to the metric[1]

$$ds^2 = \left(1 + \frac{k^2Q^2}{r^4}\right)\left(1 - \frac{2m}{r} + \frac{Q^2}{2r^2}\right)dt^2 - \left(1 - \frac{2m}{r} + \frac{Q^2}{2r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ , \tag{2.2}$$

where m and Q are the mass and charge of the particle in the units $G = c = 1$. In restored units, they have the values

$$m = GM/c^2 \ , \tag{2.3}$$

$$Q^2 = 8\pi G e^2/c^4 \ , \tag{2.4}$$

where M and e are the real mass and charge of the object. We note that when $k \to 0$, the metric (2.2) simply becomes the Reissner-Nordström[3] solution. Further, it displays the usual coordinate singularity at

$$r = m + (m^2 - \tfrac{1}{2}Q^2)^{1/2} \ . \tag{2.5}$$

Examination of the metric shows that the constant k enters with the electric charge, so that it will make itself felt only in the presence of charged objects. For example, suppose we consider the bending of light about a massive charged body. Following the usual approach to the problem,[4] we attempt to find the solution to the variational equation

$$\delta \int g_{\mu\nu} \frac{dx^\mu}{dq} \frac{dx^\nu}{dq} dq = 0 \ . \tag{2.6}$$

We specialize to the case where the light ray is restricted to the equatorial plane, so that $\theta = \frac{1}{2}\pi$ and $\dot\theta = 0$ (where the dot refers to differentiation with respect to q). Our variational equation then leads to the conditions

$$\frac{d}{dq}(r^2\dot\phi) = 0 \tag{2.7}$$

and

$$\frac{d}{dq}\left[\left(1 + \frac{k^2Q^2}{r^4}\right)\left(1 - \frac{2m}{r} + \frac{Q^2}{2r^2}\right)\dot t\,\right] = 0 \ . \tag{2.8}$$

Thus,

$$r^2\dot\phi = \text{constant} \equiv h \tag{2.9}$$

and

$$\left(1 + \frac{k^2Q^2}{r^4}\right)\left(1 - \frac{2m}{r} + \frac{Q^2}{2r^2}\right)\dot t = \text{constant} \equiv l \ . \tag{2.10}$$

Since light follows a null geodesic, that is,

$$\left(1 + \frac{k^2Q^2}{r^4}\right)\left(1 - \frac{2m}{r} + \frac{Q^2}{2r^2}\right)\dot t^2 - \left(1 - \frac{2m}{r} + \frac{Q^2}{2r^2}\right)^{-1}\dot r^2 - r^2\dot\phi^2 = 0 \ , \tag{2.11}$$

then Eqs. (2.9) and (2.10) imply that

$$l^2 - \left(1 + \frac{k^2Q^2}{r^4}\right)\dot r^2 - \frac{h^2}{r^2}\left(1 + \frac{k^2Q^2}{r^4}\right)\left(1 - \frac{2m}{r} + \frac{Q^2}{2r^2}\right) = 0 \ . \tag{2.12}$$

We define a function $u(\phi)$ by

$$u(\phi) = \frac{1}{r(\phi)} \ , \tag{2.13}$$

so that

$$\dot r = -hu' \ , \tag{2.14}$$

where the prime denotes differentiation with respect to ϕ. Equation (2.12) becomes

$$-l^2 + (1 + k^2Q^2u^4)h^2u'^2 + h^2u^2(1 + k^2Q^2u^4)(1 - 2mu + \tfrac{1}{2}Q^2u^2) = 0 \ . \tag{2.15}$$

Differentiating with respect to ϕ yields

$$u'[2k^2Q^2u^3u'^2 + u''(1 + k^2Q^2u^4) + u - 3mu^2 + Q^2u^3 + 3k^2Q^2u^5 - 7mk^2Q^2u^6 + 2k^2Q^4u^7] = 0 \ . \tag{2.16}$$

In the limit $k \to 0$, this becomes the differential equation obtained from the Reissner-Nordström solution

$$u'[u'' + u - 3mu^2 + Q^2u^3] = 0 \ . \tag{2.17}$$

Comparing Eqs. (2.16) and (2.17), we see that k enters as a "correction" term smaller by $1/r^3$ than the mass correction term. Hence, it would be expected that the effect of k upon light bending would be extremely difficult to measure.

Let us pursue another example, the gravitational frequency shift of spectral lines emitted by a charged object. Let us denote by a subscript s the mass, charge, and radius of the source (assuming it is spherical), and similarly use a subscript d for those characteristics of the detector. Then, the proper time between wave fronts, as measured by the source, is

$$d\tau_s = \left[\left(1 + \frac{k^2Q_s^{\,2}}{r_s^{\,4}}\right)\left(1 - \frac{2m_s}{r_s} + \frac{Q_s^{\,2}}{2r_s^{\,2}}\right)\right]^{1/2} dt \ . \tag{2.18}$$

Similarly, for the detector it is

$$d\tau_d = \left[\left(1+\frac{k^2 Q_d{}^2}{r_d{}^4}\right)\left(1-\frac{2m_d}{r_d}+\frac{Q_d{}^2}{2r_d{}^2}\right)\right]^{1/2} dt \ . \tag{2.19}$$

If the frequency of emission of the signal is ν_s, then the measured frequency shift $\Delta\nu$ is given by

$$\frac{\Delta\nu}{\nu_s} = \frac{\nu_s \Delta\tau_s/\Delta\tau_d - \nu_s}{\nu_s} = \left[\frac{\left(1+\frac{k^2Q_s{}^2}{r_s{}^4}\right)\left(1-\frac{2m_s}{r_s}+\frac{Q_s{}^2}{2r_s{}^2}\right)}{\left(1+\frac{k^2Q_d{}^2}{r_d{}^4}\right)\left(1-\frac{2m_d}{r_d}+\frac{Q_d{}^2}{2r_d{}^2}\right)}\right]^{1/2} - 1 \ . \tag{2.20}$$

To see the effects of k, let us, for simplicity, assume that the masses of the emitter and detector, as well as the charge of the detector, are negligible. Then, Eq. (2.20) becomes

$$\left(\frac{\Delta\nu}{\nu_s}\right)_Q = \left[\left(1+\frac{k^2Q^2}{R^4}\right)\left(1+\frac{Q^2}{2R^2}\right)\right]^{1/2} - 1 \ , \tag{2.21}$$

where Q and R are now the charge and radius of the source. Expanding the square root and retaining terms up to R^{-4}, we find

$$\left(\frac{\Delta\nu}{\nu_s}\right)_Q = \frac{Q^2}{4R^2} + \frac{k^2Q^2}{2R^4} - \frac{Q^4}{32R^4} \ . \tag{2.22}$$

Assuming that $Q<R$, then this relation shows that the frequency will be blue-shifted, with the contribution from k being $O(1/R^2)$ in comparison to the shift obtained from the simple Reissner-Nordström solution.

III. MAGNETIC FIELD SOLUTIONS

In our previous paper, we also examined the equations of motion generated by our field equations, using the methods of Einstein, Infeld, and Hoffmann.[5] We found that Eqs. (1.5)–(1.7) resulted in the equation of motion

$${}^1m\,\frac{d^2\vec{r}}{dt^2} = -\frac{{}^1m\,{}^2m\,\vec{r}}{r^3} + \frac{{}^1e\,{}^2e\,\vec{r}}{r^3} \ . \tag{3.1}$$

Here, the superscripts 1 and 2 are particle labels, and r is the distance between the particles. Now e is conventionally taken to be the electronic charge, and we see that Eq. (3.1) then predicts repulsion between like charges. However, e could also be the field strength of a magnetic monopole, Eq. (3.1) then being an expression of the repulsion between magnetic sources of like sign. We must now find out whether our field equations allow these monopoles, and, if they do, what it implies about the value of k.

We write our metric for the pure magnetic case in the form

$$g_{\mu\nu} = \begin{bmatrix} -\alpha & 0 & 0 & 0 \\ 0 & -\beta & f\sin\theta & 0 \\ 0 & -f\sin\theta & -\beta\sin^2\theta & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix} \ . \tag{3.2}$$

Equation (1.4) leads to the contravariant tensor

$$g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & \frac{-\beta}{(f^2+\beta^2)} & \frac{f}{(f^2+\beta^2)\sin\theta} & 0 \\ 0 & \frac{-f}{(f^2+\beta^2)\sin\theta} & \frac{-\beta}{(f^2+\beta^2)\sin^2\theta} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} \end{bmatrix} \ . \tag{3.3}$$

We find the affine connection from Eq. (1.1). This has been done by previous authors,[6] and we simply quote their results here:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{\alpha'}{2\alpha} \ , \\
\Gamma^1_{22} &= \csc^2\theta\,\Gamma^1_{33} = \frac{fB-\beta A}{2\alpha} \ , \\
\Gamma^1_{23} &= -\Gamma^1_{32} = \frac{fA+\beta B}{2\alpha}\sin\theta \ , \\
\Gamma^1_{44} &= \frac{\gamma'}{2\alpha} \ , \\
\Gamma^2_{12} &= \Gamma^2_{21} = \tfrac{1}{2}A \ , \\
\Gamma^2_{13} &= -\Gamma^2_{31} = \tfrac{1}{2}B\sin\theta \ , \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \ , \\
\Gamma^3_{13} &= \Gamma^3_{31} = \tfrac{1}{2}A \ , \\
\Gamma^3_{12} &= -\Gamma^3_{21} = -\tfrac{1}{2}B\csc\theta \ , \\
\Gamma^3_{23} &= \Gamma^3_{32} = \cot\theta \ , \\
\Gamma^4_{14} &= \Gamma^4_{41} = \frac{\gamma'}{2\gamma} \ ,
\end{aligned} \tag{3.4}$$

where

$$A = \frac{ff'+\beta\beta'}{f^2+\beta^2} \ , \qquad B = \frac{f\beta'-\beta f'}{f^2+\beta^2} \ . \tag{3.5}$$

Here, the primes denote differentiation with respect to r. From the expression (1.2) for $R_{\mu\nu}$, we find

$$R_{11} = -A' - \tfrac{1}{2}(A^2 + B^2) + A\frac{\alpha'}{2\alpha} + \Gamma^4_{14}\left(\frac{\alpha'}{2\alpha} - \Gamma^4_{14}\right) - \Gamma^4_{14,1}\ , \tag{3.6}$$

$$R_{22} = \csc^2\theta\, R_{33} = [(fB - \beta A)/2\alpha]' + (fB - \beta A)[\ln(\alpha\gamma)]'/4\alpha + B(fA + \beta B)/2\alpha + 1\ , \tag{3.7}$$

$$R_{44} = \Gamma^1_{44,1} + \Gamma^1_{44}\left(\frac{\alpha'}{2\alpha} - \Gamma^4_{14} + A\right)\ , \tag{3.8}$$

$$\csc\theta\, R_{23} = -\csc\theta\, R_{32} = [(fA + \beta B)/2\alpha]' - B(fB - \beta A)/2\alpha + (fA + \beta B)(\alpha' + 2\alpha\Gamma^4_{14})/4\alpha^2\ . \tag{3.9}$$

Similarly, we find from Eq. (1.9) that

$$I_{11} = \frac{1}{2k^2}\,\alpha\,\frac{f^2}{(f^2 + \beta^2)}\ , \tag{3.10}$$

$$I_{22} = \csc^2\theta\, I_{33} = -\frac{1}{2k^2}\frac{\beta f^2}{f^2 + \beta^2}\ , \tag{3.11}$$

$$I_{44} = -\frac{1}{2k^2}\,\gamma\,\frac{f^2}{f^2 + \beta^2}\ , \tag{3.12}$$

$$\csc\theta\, I_{23} = -\csc\theta\, I_{32} = -\frac{1}{2k^2} f\left(\frac{\beta^2}{f^2 + \beta^2} + 1\right)\ . \tag{3.13}$$

Combining Eqs. (3.6), (3.8), and (3.4) with Eq. (1.6), we obtain the relations

$$-A' - \tfrac{1}{2}(A^2 + B^2) + A\frac{\alpha'}{2\alpha} + \frac{\gamma'}{2\gamma}\left(\frac{\alpha'}{2\alpha} - \frac{\gamma'}{2\gamma}\right) - \left(\frac{\gamma'}{2\gamma}\right)' = -I_{11}\ , \tag{3.14}$$

$$\left(\frac{\gamma'}{2\alpha}\right)' + \frac{\gamma'}{2\alpha}\left(\frac{\alpha'}{2\alpha} - \frac{\gamma'}{2\gamma} + A\right) = -I_{44}\ . \tag{3.15}$$

Solving Eq. (3.15) for γ'', we find

$$\frac{\gamma''}{2\gamma} = \frac{\alpha'\gamma'}{4\alpha\gamma} + \left(\frac{\gamma'}{2\gamma}\right)^2 - \frac{\gamma'}{2\gamma}A - \frac{\alpha}{\gamma}I_{44}\ . \tag{3.16}$$

Substituting Eq. (3.16) into (3.14) and simplifying somewhat, we find

$$-A' - \tfrac{1}{2}(A^2 + B^2) + (\tfrac{1}{2}A)[\ln(\alpha\gamma)]' = -\left(I_{11} + \frac{\alpha}{\gamma}I_{44}\right)\ . \tag{3.17}$$

But Eqs. (3.10) and (3.12) show that the right-hand side of the above equation vanishes. Further, since there are no 1, 4 cross terms, then we expect $\alpha\gamma = 1$. Hence we find

$$-A' - \tfrac{1}{2}(A^2 + B^2) = 0\ . \tag{3.18}$$

We now substitute a monopole field of the form

$$H = \frac{l}{r^2}\ , \tag{3.19}$$

where l is a measure of the magnetic charge.

Similarly to the electric field case, Eqs. (1.12) and (1.13) show that

$$f\sin\theta = kr^2\sin\theta\,\frac{l}{r^2} \tag{3.20}$$

Equation (3.18) then becomes

$$k^2l^2\beta\beta'' + \beta^3\beta'' - \tfrac{1}{2}\beta^2\beta'^2 + \tfrac{3}{2}k^2l^2\beta'^2 = 0\ . \tag{3.21}$$

We substitute a solution for β of the form

$$\beta = \left(a + \frac{b}{r^2} + \frac{c}{r^4}\right)r^2\ , \tag{3.22}$$

where odd powers of r vanish, and obtain from Eq. (3.21) a polynomial in powers of r from r^4 to r^{-10}. The condition that this equation holds for all r leads to eight equations for the coefficients, four of which are

$$ab = 0\ ,\quad 2k^2l^2 + 3ac + b^2 = 0\ ,\quad bc = 0\ , \tag{3.23}$$

and

$$c = 0\ .$$

The only nontrivial solution for β is then $\beta = ar^2$. We see that this implies that the product kl vanishes. As stated above, our field equations yield Maxwell's equations in the $k \to 0$ limit, and these equations do not admit magnetic monopoles. For $k \neq 0$, l must vanish, and so too the magnetic monopoles. The set of electromagnetic equations

$$F^{\mu\nu}{}_{;\nu} = J^\mu_E$$

and (3.24)

$${}^*F^{\mu\nu}{}_{;\nu} = J^\mu_M\ ,$$

where the asterisk denotes the tensor dual to $F^{\mu\nu}$ and J^μ_E, J^μ_M are the electric and magnetic current densities, respectively, are incompatible with our solutions.

IV. CONCLUSION

We have applied our solution for a static spherically symmetric electric field in the nonsymmetric unified field theory to two physical problems: the bending of light about a charged object and the gravitational blue-shift of light emitted by a charged object. In both cases the universal constant enters as an $O(1/r^2)$ correction to the predictions of the Reissner-Nordström solution. Thus, it will be of importance for large charge-to-distance ratios.

We have also investigated the predictions of the theory as they affect magnetic monopoles. Although these monopoles are allowed by certain theories, we find here that no value of k admits their existence.

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