Physical interpretation of and light propagation in the nonsymmetric unified field theory[1]

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The geometric problems of unified theories of gravitation and electromagnetism which contain a nonsymmetric metric tensor are investigated. Although many of the familiar geometric properties of the Einstein–Maxwell theory are no longer valid, there do exist autoparallel 'geodesics' or paths, whose tangent vectors have constant length. It is shown in the weak field approximation that the theory allows for two oscillatory solutions, corresponding to electromagnetic and gravitational waves. The propagation of these waves is investigated in the geometric optics approximation. To the approximations used, light travels on null geodesics of the gravitational background in this theory, although not all relevant cases are solved.

On étudie les problèmes géométriques des théories unifiées de la gravitation et de l'électromagnétisme qui contiennent un tenseur métrique non symétrique. Alors que plusieurs des propriétés géométriques familières de la théorie Einstein–Maxwell ne sont plus valides, et existe des "géodésiques" ou parcours autoparallèles dont les vecteurs tangents ont une longueur constante. On montre dans l'approximation de champ faible que la théorie permet deux solutions oscillatoires correspondant aux ondes électromagnétiques et gravitationnelles. La propagation de ces ondes est étudiée à l'approximation de l'optique géométrique. Dans les limites des approximations utilisées dans cette théorie, la lumière se propage le long des géodésiques nulles du fond gravitationnel; tous les cas pertinents ne sont cependant pas résolus.

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1. Introduction

We have proposed (1, 2) that the difficulties associated with Einstein's formulation (3, 4) of the nonsymmetric[3] unified field theory can be eliminated by a small modification of Einstein's Lagrangian. In retrospect, a mathematically equivalent modification had been proposed by Bonnor (5), although the physical interpretation was different. In this theory, the Lagrangian density becomes

$$[1.1]\quad \mathscr{L} = \sqrt{-g}\, g^{\mu\nu}B_{\mu\nu} + (4\pi/k^2)\sqrt{-g}\, g^{\mu\nu}g_{[\nu\mu]}$$

where g is the determinant of the nonsymmetric metric tensor $g_{\mu\nu}$. Here, k is a universal constant with dimensions of length, and $B_{\mu\nu}$, the contracted curvature tensor, is given in terms of the nonsymmetric connection $L^{\alpha}{}_{\mu\nu}$ by

$$[1.2]\quad B_{\mu\nu} = \partial_\alpha L^{\alpha}{}_{\mu\nu} - \tfrac{1}{2}(\partial_\nu L^{\alpha}{}_{\mu\alpha} + \partial_\mu L^{\alpha}{}_{\nu\alpha}) - L^{\alpha}{}_{\mu\sigma}L^{\sigma}{}_{\alpha\nu} + L^{\alpha}{}_{\mu\nu}L^{\sigma}{}_{\alpha\sigma}$$

Variation of the action integral gives rise to a set of field equations. Unfortunately, the connection $L^{\alpha}{}_{\mu\nu}$ is underdetermined by the field equation relating $L^{\alpha}{}_{\mu\nu}$ and $g_{\mu\nu}$ (6, 7). To circumvent this problem, a projective transformation which leaves the autoparallel paths unchanged (8) is applied to generate a new connection $\Gamma^{\alpha}{}_{\mu\nu}$:

$$[1.3]\quad \Gamma^{\alpha}{}_{\mu\nu} = L^{\alpha}{}_{\mu\nu} + \delta^{\alpha}{}_{\mu}V_{\nu}$$

where

$$[1.4]\quad V_\nu = \tfrac{1}{3}(L^{\sigma}{}_{\nu\sigma} - L^{\sigma}{}_{\sigma\nu})$$

The field equations, in terms of the new connection, are:

$$[1.5]\quad \partial_\alpha g_{\mu\nu} - g_{\mu\sigma}\Gamma^{\sigma}{}_{\alpha\nu} - g_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\alpha} = 0$$

$$[1.6]\quad \Gamma^{\alpha}{}_{[\mu\alpha]} = 0$$

$$[1.7]\quad R_{[\mu\nu]} + I_{[\mu\nu]} = V_{\mu,\nu} - V_{\nu,\mu}$$

$$[1.8]\quad R_{(\mu\nu)} + I_{(\mu\nu)} = 0$$

$$[1.9]\quad I_{\mu\nu} = -(4\pi/k^2)(g_{\mu\sigma}g^{[\sigma\rho]}g_{\rho\nu} + \tfrac{1}{2}g_{\mu\nu}g_{[\sigma\rho]}g^{[\sigma\rho]} + g_{[\mu\nu]})$$

where R is the same function of Γ as B is of L in [1.2]. The quantity V, which, from [1.7], is a potential, is not determined by the field equations so that [1.7] is written as a curl

$$[1.10]\quad {}_{[}(R_{[\mu\nu]} + I_{[\mu\nu]})_{,\sigma]} = 0$$

Equation [1.6] is equivalent to the condition

$$[1.11]\quad \partial_\alpha(\sqrt{-g}\, g^{[\mu\alpha]}) = 0$$

In the static spherically symmetric case, these

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[3]We will use the notation $g_{(\mu\nu)} = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})$ and $g_{[\mu\nu]} = \frac{1}{2}(g_{\mu\nu} - g_{\nu\mu})$ The convention $g^{\mu\alpha}g_{\nu\alpha} = \delta^{\mu}{}_{\nu}$ will be used to define $g^{\mu\nu}$ A comma will denote coordinate differentiation.

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equations can be solved exactly to yield the line element

$$[1.12]\quad ds^2 = \left(1 + \frac{k^2Q^2}{r^4}\right)\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

in spherical polar coordinates, where m and Q are the mass and charge, respectively, in units where $G = c = 1$. The only nonvanishing element of $g_{[\mu\nu]}$ is

$$[1.13]\quad g_{[14]} = kQ/r^2$$

In the limit $k \to 0$, [1.12] reduces to the Reissner–Nordström solution (9, 10), suggesting that our field equations should reduce to the Einstein–Maxwell field equations in this limit. Equation [1.13] suggests the identification with the generalized electromagnetic field tensor $F_{\mu\nu}$:

$$[1.14]\quad g_{[\mu\nu]} = kF_{[\mu\nu]}$$

Applying the $k \to 0$ limit [1.5], [1.7], [1.8], and [1.11] yield

$$[1.15]\quad R_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$[1.16]\quad -(8\pi/k)F_{\mu\nu} = V_{\mu,\nu} - V_{\nu,\mu}$$

$$[1.17]\quad \partial_\alpha(\sqrt{-g}\,F^{\mu\alpha}) = 0$$

$$[1.18]\quad T_{\mu\nu} = F_\mu{}^\alpha F_{\alpha\nu} + \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

and

$$[1.19]\quad g_{\mu\nu;\alpha} = 0$$

where $R_{\mu\nu}$ and $g_{\mu\nu}$ are now symmetric and where the semicolon indicates the covariant derivative. Equation [1.16] suggests the identification (in the $k \to 0$ limit):

$$[1.20]\quad V_\mu = -(8\pi/k)A_\mu$$

where A_μ is the usual electromagnetic potential.

We must note two things about $F_{\mu\nu}$ and A_μ. First, although $F_{\mu\nu}$ has the same form for all k, it obeys Maxwell's equations only when $k = 0$. Second, A_μ reduces to the electromagnetic potential only when $k = 0$. As defined in [1.20], A_μ is in general a function of k, and in the static spherically symmetric case, has the form

$$[1.21]\quad \begin{aligned} A_i &= 0 \quad (i = 1, 2, 3) \\ A_4 &= Q/r - k^2Qm/8\pi r^4 - k^2Q^3/2r^5 \end{aligned}$$

Thus, the electromagnetic potential of the Einstein–Maxwell theory gains correction terms of order k^2.

The field equations given above have been shown (11) to give the correct equations of motion to the lowest nontrivial order of approximation. Further properties of the theory can be found in work by Boal (12), Borchsenius (13), Moffat (14), Kunstatter and Moffat (15), and references cited therein.

2. Physical Interpretation

The geometric aspects of the Einstein–Maxwell theory are well known. One can construct locally flat frames which retain their 'flatness' as they are parallel propagated along a geodesic. The geodesics themselves are the same whether they are generated from the symmetric connection of the field equation [1.19] or from the connection found from the extremal of arc length since the two connections are identical. Further, the length of a vector or its product with another vector or its angle with that vector all remain the same when parallel transported along a geodesic. Since the purpose of the unified field theory was to incorporate electromagnetism into the geometry of space–time as had been done with gravity, we must determine how much of the beautiful geometric structure of the Einstein–Maxwell theory still exists.

We can see immediately that in the unified theory things are not as straightforward. Not only do we have two different connections entering into the field equations, $L^\alpha{}_{\mu\nu}$ and $\Gamma^\alpha{}_{\mu\nu}$, but we also have a third connection generated by demanding that the arc length must be extremal,

$$[2.1]\quad \delta\int ds = 0$$

namely the Levi–Civita symmetric connection $\Lambda^\alpha{}_{\mu\nu}$:

$$[2.2]\quad g_{(\alpha\sigma)}\Lambda^\alpha{}_{\mu\nu} = \tfrac{1}{2}(g_{(\mu\alpha),\nu} + g_{(\alpha\nu),\mu} - g_{(\mu\nu),\alpha})$$

The first two connections mentioned above are related by a projective transformation, so any vector directions parallel with respect to $L^\alpha{}_{\mu\nu}$ are parallel with respect to $\Gamma^\alpha{}_{\mu\nu}$. Hence, the paths (which we shall use instead of the word geodesic) of the autoparallel tangent vectors are the same (8). However, the connection $\Lambda^\alpha{}_{\mu\nu}$ is not related to the other two by a projective transformation (not even their symmetric parts are so related) so that a completely different set of paths is generated

We will return to the Levi–Civita connection later in this section. Let us first look at the connections which arise in the field equations of the theory with a real nonsymmetric metric tensor and consider the product $g_{\mu\nu}A^\mu B^\nu$ (since $g_{\mu\nu}$ is nonsymmetric, the order of the vectors in the product is important) as it is parallel transported along some path generated by a nonsymmetric connection $W^\alpha{}_{\mu\nu}$. The variation of the product with respect to an affine parameter t along the path is:

$$[2.3]\quad \frac{\delta}{\delta t}(g_{\mu\nu}A^{\mu}B^{\nu}) = (g_{\mu\nu,\alpha} - g_{\sigma\nu}W^{\sigma}{}_{\mu\alpha} - g_{\mu\sigma}W^{\sigma}{}_{\nu\alpha})k^{\alpha}A^{\mu}B^{\nu}$$

where k^{α} is the vector tangent to the curve. Let us ask that this variation vanish in analogy with general relativity. Now, if $W^{\alpha}{}_{\mu\nu}$ be identified with the connection $\Gamma^{\alpha}{}_{\mu\nu}$, then the field equation [1.5] implies that $\Gamma^{\alpha}{}_{[\mu\nu]} = 0$, so that the theory reduces to general relativity. If the connection $L^{\alpha}{}_{\mu\nu}$ is chosen instead, then again we find that $L^{\alpha}{}_{[\mu\nu]} = 0$. Clearly, this implies that the potential V also has to vanish. Thus, we see that the product of two arbitrary vectors is not preserved by parallel transport. If we choose $A^{\mu} = B^{\mu}$, then the same situation arises so that lengths are not, in general, preserved.

However, if we put $B^{\nu} = k^{\nu}$, then the field equation [1.5] implies that the right-hand side of [2.3] vanishes. So the product of the tangent vector with an arbitrary vector is preserved under parallel transport. We note this is only valid for the order of vectors specified, not for $g_{\mu\nu}k^{\mu}B^{\nu}$. Lastly, if we put both vectors equal to the tangent vector, then we find that its length is invariant. In summary, we have a minimal geometry in that we can define paths whose tangent vectors have both constant length and can form a product with an arbitrary vector. However, the product of two arbitrary vectors, and hence the angle between them, is not invariant, nor is the length of an arbitrary vector. Indeed, a timelike vector can be turned into a spacelike one. As an example of this consider parallel transport of a null vector along a radial geodesic in the exact spherically symmetric solution mentioned above. Evaluating the derivative of its length at the initial point, we find it is non-zero. Therefore on one side the vector is timelike, on the other it is spacelike.

An alternative approach to the theory which we considered earlier was to allow $g_{\mu\nu}$ to be Hermitian (1), so that the antisymmetric part is pure imaginary. The exact solution to the field equations for the spherically symmetric case was shown to have the unusual property of becoming Euclidian within a sphere of radius $\sqrt{kQ}$. Some of the properties of this solution are presently being investigated (15).

If $g_{\mu\nu}$ is Hermitian, then [1.5] implies that $\Gamma^{\alpha}{}_{\mu\nu}$ must also be Hermitian in the lower two indices. From [1.7] it follows that the potential V^{μ} is pure imaginary. If we combine these two results with [1.3] then we find that the connection $L^{\alpha}{}_{\mu\nu}$ is neither Hermitian nor generally complex. For example, $L^{\alpha}{}_{\beta\beta}$ $(\alpha \neq \beta)$ is real whereas $L^{\beta}{}_{\beta\beta}$ is complex. Hence, we find that the Hermiticity of $\Gamma^{\alpha}{}_{\mu\nu}$ has entered in a somewhat artificial way.

If we use this Hermitian metric to form a product with two real vectors, as we did before with the real nonsymmetric metric, then none of our conclusions are changed. If we extend our considerations to complex vectors, then we can define a product which is invariant, namely $g_{\mu\nu}A^{\mu}\bar{B}^{\nu}$, where the bar indicates complex conjugation. The product is propagated in such a way that the vectors are propagated independently, then the complex conjugation operator is applied. This has the effect of rearranging the covariant indices of $W^{\alpha}{}_{\mu\nu}$ in [2.3] to match those in [1.5]. This has not significantly improved the geometrical problems because we are now faced with the situation where the angle between two vectors is complex and an arbitrary real vector (the tangent vector being an exception) will become complex as it is parallel propagated along a curve (the length of a vector will remain real). However, this product operation is only successful if the parallel propagation occurs along a real vector; for the general complex case it is no longer useful. Because of these associated problems we will drop the Hermitian metric from our discussions.

We should stop to consider why there is so much trouble in dealing with anything but tangent vectors. The answer lies in the incompatibility of these connections with a locally flat frame which is invariant under parallel transport. In the Einstein–Maxwell theory, if there exists a coordinate system such that $g_{\mu\nu} = \eta_{\mu\nu}$ (the Minkowski metric), then the connection vanishes (in the Cartesian coordinate system). Here, having a locally flat metric with $g_{(\mu\nu)} = \eta_{\mu\nu}$ only implies that

$$[2.4]\quad \eta_{\mu\sigma}\Gamma^{\sigma}{}_{(\alpha\nu)} + \eta_{\nu\sigma}\Gamma^{\sigma}{}_{(\alpha\mu)} + \eta_{\alpha\sigma}\Gamma^{\sigma}{}_{(\mu\nu)} = 0$$

and consequently

$$[2.5]\quad 2\eta_{i\sigma}\Gamma^{\sigma}{}_{(i\alpha)} + \eta_{\alpha\sigma}\Gamma^{\sigma}{}_{(ii)} = 0$$

and

$$[2.6]\quad \Gamma^{i}{}_{ii} = 0$$

where the i's are not summed in [2.5] and [2.6].

Since the symmetric parts of the connections do not vanish, then neither does the second derivative term in the equation of motion

$$[2.7]\quad \frac{d^2x^{\alpha}}{dp^2} + \Gamma^{\alpha}{}_{\mu\nu}\frac{dx^{\mu}}{dp}\frac{dx^{\nu}}{dp} = 0$$

Thus, we cannot construct a true Lorentz frame.

A solution to these problems can be found by introducing yet a third connection, the Levi–Civita connection defined by [2.2]. This is not particularly elegant, as the theory will then have three connections, two of which appear in the derivation of the field equations and one of which defines parallel propagation. The motivation for introducing the curvature tensor, which is not defined in terms of $\Lambda^{\alpha}{}_{\mu\nu}$, appears to be lost.

However, $\Lambda^{\alpha}{}_{\mu\nu}$ does vanish when $g_{(\mu\nu)} = \eta_{\mu\nu}$ and so we can construct local Lorentz frames. Furthermore, if we define the product of the vectors by $g_{(\mu\nu)}A^{\mu}B^{\nu}$, then this product is invariant under parallel propagation with respect to $\Lambda^{\alpha}{}_{\mu\nu}$.

In summary, we have two cases. We can use $\Gamma^{\alpha}{}_{\mu\nu}$ to define parallel propagation, in which case we have a motivation for the theory but can only properly incorporate tangent vectors into the geometry. On the other hand, we can use $\Lambda^{\alpha}{}_{\mu\nu}$ for parallel propagation, but what we gain in widening the geometry considerably we lose in motivation for the theory. One possible way of distinguishing between these two connections is to find out along which, if either, light chooses to propagate.

Unfortunately, present day experiment is of no great help either in sorting this problem out or, for that matter, in testing the theory. Because present day experiments are done using massive uncharged objects, then we can test only the $Q = 0$ solution of this theory. This, of course, is the same as general relativity, and so all of the predictions of this theory, such as the parameters of the parametrized post-Newtonian (PPN) formalism, are the same as in general relativity. Since the corrections (2) in the $Q \neq 0$ case given here are several powers of r down from the Einstein–Maxwell corrections, it is unlikely that experiment will test this theory for some time to come, as even Einstein–Maxwell theory remains untested.

3. Weak Field Approximation

In order to study light propagation in this theory, we must first establish that light exists as distinct from gravitational radiation. As a first step we will show that in the weak field limit electromagnetic and gravitational null plane waves analogous to those of the Einstein–Maxwell theory are allowed solutions. Because of the extra terms in the nonsymmetric theory, it is not immediately obvious that these two types of waves still can be independently present.

In the weak field approximation, the metric tensor $g_{\mu\nu}$ is expanded in a series with the flat space-time metric $\eta_{\mu\nu}$ as the principal term and a sum over progressively weaker perturbing fields:

[3.1] $$g_{\mu\nu} = \eta_{\mu\nu} + \lambda g'_{\mu\nu} + \lambda^2 g''_{\mu\nu} + \ldots$$

where the number of primes indicates the order of approximation involved. In Appendix A, the field equations are expanded to second order in λ. For the present, we will assume that the perturbing fields are very weak, so that we can work in first order. Equations [1.6] to [1.8] then become

[3.2] $$\eta^{\mu\alpha} g'_{[\mu\nu],\alpha} = 0$$

[3.3] $$\tfrac{1}{2}\Box g'_{[\mu\nu]} - (8\pi/k^2) g'_{[\mu\nu]} = V'_{\mu,\nu} - V'_{\nu,\mu}$$

[3.4] $$-\Box g'_{(\mu\nu)} + \eta^{\alpha\beta}(-g'_{(\beta\alpha),\mu,\nu} + g'_{(\beta\nu),\mu,\alpha} + g'_{(\alpha\mu),\beta,\nu}) = 0$$

respectively. Clearly, the equations have separated into symmetric and antisymmetric parts. If we work in harmonic coordinates, such that

[3.5] $$g^{\alpha\beta}\Gamma^{\rho}{}_{\alpha\beta} = 0$$

then we find to first order that [3.5] gives the condition

[3.6] $$\eta^{\alpha\beta}\partial_{\alpha} g'_{(\beta\nu)} = \tfrac{1}{2}\eta^{\alpha\beta}\partial_{\nu} g'_{(\alpha\beta)}$$

For simplicity of notation, we will rewrite $g'_{[\mu\nu]}$ as $kF'_{\mu\nu}$ and V'_{μ} as $-(8\pi/k)A_{\mu}$. Then [3.2] to [3.4] have the following solution in harmonic coordinates:

[3.7] $$F'_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \qquad \Box A_{\mu} = 0 \qquad \partial_{\mu}A^{\mu} = 0$$

and

[3.8] $$\Box g'_{(\mu\nu)} = 0 \qquad \eta^{\alpha\beta}\partial_{\alpha} g'_{(\beta\mu)} = \tfrac{1}{2}\eta^{\alpha\beta}\partial_{\mu} g'_{(\alpha\beta)}$$

These are just the usual equations for electromagnetic and gravitational waves. The field equations allow the divergence of the potential A to be a constant which we choose to set equal to zero, so that the equations have the solution

[3.9] $$A_{\mu}(x) = \omega_{\mu}\exp(iq_{\alpha}x^{\alpha}) + \text{c.c.}$$

[3.10] $$g'_{(\mu\nu)}(x) = h_{\mu\nu}\exp(ip_{\alpha}x^{\alpha}) + \text{c.c.}$$

where $q(p)$ and $\omega_{\mu}(h_{\mu\nu})$ are the momentum and polarization vector (tensor), respectively, of the electromagnetic (gravitational) wave, subject to the conditions:

[3.11] $$q_{\mu}q^{\mu} = 0, \quad p_{\mu}p^{\mu} = 0 \qquad q_{\mu}\omega^{\mu} = 0, \quad p_{\mu}h^{\mu}{}_{\nu} = \tfrac{1}{2}p_{\nu}h^{\mu}{}_{\mu}$$

It is clear from [A15] to [A18] that the term corresponding to the electromagnetic stress energy tensor, [1.18], will only appear if the equations are carried to second order. In particular, $g'_{[\mu\nu]}$ will effect the solution for $g''_{(\mu\nu)}$ through [A8], [A12], and [A15]. Similarly, the first-order field $g'_{(\mu\nu)}$ will appear in [A14], which governs $g''_{[\mu\nu]}$. It should be noted that if we assume that the first-order gravitational field $g'_{(\mu\nu)}$ vanishes, then [A10], [A14], and [A18] generate the equations

$$[3.12] \qquad \eta^{\mu\alpha} g''_{[\mu\nu],\alpha} = 0$$

$$[3.13] \qquad \tfrac{1}{2}\Box g''_{[\mu\nu]} - (8\pi/k^2) g''_{[\mu\nu]} = V''_{\mu,\nu} - V''_{\nu,\mu}$$

Similarly, [A12] and [A16] give the equation

$$[3.14] \qquad \Gamma''^{\alpha}_{(\mu\nu),\alpha} - \tfrac{1}{2}(\Gamma''^{\alpha}_{(\mu\alpha),\nu} + \Gamma''^{\alpha}_{(\nu\alpha),\mu}) - \Gamma'^{\alpha}_{[\mu\sigma]}\Gamma'^{\sigma}_{[\alpha\nu]} + I''_{(\mu\nu)} = 0$$

Since $g''_{[\mu\nu]}$ obeys the same equations as $g'_{[\mu\nu]}$, in this approximation, the potential V''_{μ} obtained from $g''_{[\mu\nu]}$ also has a plane wave solution. The $I''_{(\mu\nu)}$ in [3.14] is just the usual electromagnetic energy-momentum tensor so in this approximation we have the analogous Einstein–Maxwell expression for $R_{(\mu\nu)}$ plus several correction terms of order k^2:

$$[3.15] \qquad R''_{(\mu\nu)} = -8\pi T''_{\mu\nu} + O(k^2)$$

If we use the plane wave solution [3.9], these extra terms vanish, leaving the general relativity expression.

4. Light Propagation

In order to press our study of these electromagnetic and gravitational waves further, we will make use of the geometric optics approximation. (See, for example, Isaacson (16), Gerlach (17), Tokuoka (18), and Misner *et al.* (19).) By this, we mean that we will apply to the background field a periodic disturbance of high frequency, and expand the field equations in power series of the wavelength, ε, of this disturbance, which is small. Because of the complexity of the resulting equations, we will also use the weak field approximation developed in Appendix A. Care will have to be taken in dealing with the relative magnitudes of the two expansion parameters λ and ε to ensure that the results are physical. We will discuss this point at length further on.

Before tackling the nonsymmetric case, we will look at some of the applications of the geometric optics approximation in the Einstein–Maxwell theory. For example, if we consider the case where $F_{\alpha\beta}$ contains no background or source term, that is

$$[4.1] \qquad F_{\alpha\beta} = A_{\alpha\beta} \exp(is/\varepsilon) + \text{c.c.}$$

where $A_{\alpha\beta}$ is slowly varying, then the weak field approximation can be avoided entirely.[4] One writes the vector potential A_α in terms of the polarization vector ω_α by means of

$$[4.2] \qquad A_\alpha = \tfrac{1}{2}\varepsilon\omega_\alpha \exp(is/\varepsilon) + \text{c.c.}$$

Demanding that $F^{\alpha\beta}$ be divergenceless and taking the high frequency ($\varepsilon \to 0$) limit then yields the desired results

$$[4.3] \qquad k^\alpha k_\alpha = 0$$

$$[4.4] \qquad k^\alpha \omega_\alpha = 0$$

where

$$[4.5] \qquad k_\alpha = s_{;\alpha} = s_{,\alpha}$$

It is well known that if k_α, the gradient of a scalar s, is null, then it is geodesic in general relativity.

As an introduction to the methods we will use in the nonsymmetric theory, we now turn our attention to a radiative disturbance on a general background in the Einstein–Maxwell theory (by which we mean a slowly varying solution of [1.15] to [1.20]), where both weak fields and geometric optics must be used to obtain null geodesic propagation. We write

$$[4.6] \qquad A_\alpha = A^0_\alpha - \frac{i\varepsilon\lambda}{2}\omega_\alpha \exp\left(\frac{is}{\varepsilon}\right) + \text{c.c.} + O(\lambda^2)$$

so that

$$[4.7] \qquad F_{\alpha\beta} = F^0_{\alpha\beta} + \lambda F'_{\alpha\beta} + O(\lambda^2)$$

where

$$[4.8] \qquad F'_{\alpha\beta} = \exp\left(\frac{is}{\varepsilon}\right)\omega_{[\alpha}k_{\beta]} + \text{c.c.} + O(\varepsilon)$$

Similarly

$$[4.9] \qquad g_{\alpha\beta} = g^0_{\alpha\beta} + \lambda g'_{\alpha\beta} + O(\lambda^2)$$

where we write

$$[4.10] \qquad g'_{\alpha\beta} = \exp\left(\frac{is}{\varepsilon}\right)(-\varepsilon^2 h_{\alpha\beta}) + \text{c.c.} + O(\varepsilon)$$

A few words are now appropriate to discuss the relative magnitude of λ and ε. In order to have a finite curvature tensor, we must take λ to zero at least as fast as ε:

$$[4.11] \qquad \lambda/\varepsilon \lesssim 1$$

When we rescale $\lambda \to \lambda'\varepsilon^n$ in the weak field approximation we find that the magnitude of the curvature components changes but the formal results obtained do not. Let us choose λ/ε to be of order unity, and then see what constraints we can place on $h'_{\alpha\beta}$ in

$$[4.12] \qquad g'_{\alpha\beta} = h'_{\alpha\beta} \exp(is/\varepsilon) + \text{c.c.}$$

to obtain physical results. As it stands, [4.12] will lead to a typical component of $R^\alpha{}_{\beta\gamma\delta}$ having a leading term of the form λ/ε^2, which clearly diverges if λ/ε is of order unity. Since we wish $R_{\alpha\beta}$ to remain finite when ε vanishes, we therefore write

$$[4.13] \qquad h'_{\alpha\beta} = \varepsilon^2 h_{\alpha\beta} + O(\varepsilon^3)$$

[4] M. Walker. Private communication.

where $h_{\alpha\beta}$ is of order unity in the two parameters. Although we could avoid the rescaling of $h'_{\alpha\beta}$ if we placed more constraints on n in

$$\lambda/\varepsilon^n \lesssim 1 \quad [4.14]$$

for the sake of making our results more easily comparable with others (for example,[5] Isaacson (16)) we take λ/ε to be of order unity. In what follows, we will assume that ω_α and $h_{\alpha\beta}$ may be as large as order unity (although they may be smaller) and that the ratio of their magnitudes varies depending on the strengths of the electromagnetic and gravitational sources, respectively. When both ω_α and $h_{\alpha\beta}$ are of order unity, we find that

$$T'_{\alpha\beta} = \exp(is/\varepsilon)(F^{0\gamma}{}_{\alpha}\omega_{[\gamma}k_{\beta]} + F^{0\gamma}{}_{\beta}\omega_{[\alpha}k_{\gamma]} + \tfrac{1}{2}g^0{}_{\alpha\beta}F^{0\rho\sigma}\omega_{[\rho}k_{\sigma]}) + \text{c.c.} + O(\varepsilon) \quad [4.15]$$

$$\Gamma'^{\alpha}{}_{\beta\gamma} = \tfrac{1}{2}g^{0\alpha\delta}\exp(is/\varepsilon)(-i\varepsilon)(h_{\beta\delta}k_\gamma + h_{\delta\gamma}k_\beta - h_{\beta\gamma}k_\delta) + \text{c.c.} + O(\varepsilon^2) \quad [4.16]$$

$$R'_{\alpha\beta} = \tfrac{1}{2}g^{0\gamma\delta}\exp(is/\varepsilon)(h_{\alpha\delta}k_\beta k_\gamma + h_{\delta\beta}k_\alpha k_\gamma - h_{\alpha\beta}k_\delta k_\gamma - h_{\delta\gamma}k_\alpha k_\beta) + \text{c.c.} + O(\varepsilon) \quad [4.17]$$

The divergence condition on $F'^{\alpha\beta}$ now gives

$$g^{0\alpha\beta}k_\alpha k_\beta = 0 \quad [4.18]$$

$$g^{0\alpha\beta}k_\alpha\omega_\beta = 0 \quad [4.19]$$

and k_α is geodesic.

Einstein's equation is complicated, having electromagnetic source terms, and is not required for our purposes here. With similar work, it can be seen that, if $h_{\alpha\beta}$ is of order unity and ω_α of order ε, gravitational and electromagnetic waves travel freely and independently along null geodesics. If $h_{\alpha\beta}$ is of order unity and ω_α of order ε^2, the curvature equation solves essentially as it did in the linearized theory (k_α null with respect to $g^{0\alpha\beta}$), but the electromagnetic divergence equation is complicated by terms in $h_{\alpha\beta}$.

Turning now to the nonsymmetric theory, we have already shown that we have the usual electromagnetic and gravitational waves for a flat space background. We now wish to consider two extensions of this. We will deal first with a gravitational background only, and then with a general background.

For a purely gravitational background, we set $g^0{}_{[\mu\nu]} = 0$, so that

$$g_{\mu\nu} = g^0{}_{(\mu\nu)} + \lambda g'_{\mu\nu} + O(\lambda^2) \quad [4.20]$$

$$\Gamma^\alpha_{\mu\nu} = \Gamma^{0\alpha}{}_{(\mu\nu)} + \lambda\Gamma'^\alpha{}_{\mu\nu} + O(\lambda^2) \quad [4.21]$$

$$R_{\mu\nu} = R^0{}_{(\mu\nu)} + \lambda R'_{\mu\nu} + O(\lambda^2) \quad [4.22]$$

If we go through the same operations as described in Appendix A for the Minkowski background, then we find that $R^0{}_{(\mu\nu)}$ and $R'_{(\mu\nu)}$ have the same form as a function of $g^0{}_{(\mu\nu)}$ and $g'_{(\mu\nu)}$ as the corresponding quantities have in general relativity. Further, $I^0{}_{(\mu\nu)}$ and $I'_{(\mu\nu)}$ both vanish, and so the equations governing gravitational waves are the same in this case as they are in general relativity. Since the results for gravitational waves are well known (16, 19) we will not discuss them further. The antisymmetric quantities are of more concern:

$$\Gamma'^\alpha{}_{[\beta\gamma]} = \tfrac{1}{2}g^{0(\alpha\delta)}(g'_{[\beta\delta],\gamma} + g'_{[\delta\gamma],\beta} + g'_{[\beta\gamma],\delta} - 2g'_{[\beta\rho]}\Gamma^{0\rho}{}_{(\gamma\delta)} - 2g'_{[\rho\gamma]}\Gamma^{0\rho}{}_{(\delta\beta)}) \quad [4.23]$$

The tracelessness condition on this connection gives

$$g^{0(\gamma\delta)}(g'_{[\alpha\delta],\gamma} - g'_{[\alpha\rho]}\Gamma^{0\rho}{}_{(\delta\gamma)} - g'_{[\rho\delta]}\Gamma^{0\rho}{}_{(\alpha\gamma)}) = 0 \quad [4.24]$$

Since

$$I'_{[\alpha\beta]} = -(8\pi/k^2)g'_{[\alpha\beta]} \quad [4.25]$$

the equation in $R'_{[\alpha\beta]}$ reads:

$$_{[}(\Gamma'^\gamma{}_{[\alpha\beta],\gamma} - \Gamma^{0\gamma}{}_{(\alpha\delta)}\Gamma'^\delta{}_{[\gamma\beta]} + \Gamma^{0\gamma}{}_{(\beta\delta)}\Gamma'^\delta{}_{[\gamma\alpha]} + \Gamma^{0\delta}{}_{(\gamma\delta)}\Gamma'^\gamma{}_{[\alpha\beta]} - (8\pi/k^2)g'_{[\alpha\beta]})_{,\rho]} = 0 \quad [4.26]$$

As before, we now introduce a high frequency perturbation in $g'_{[\alpha\beta]}$:

$$g'_{[\alpha\beta]} = \exp(is/\varepsilon)(-\varepsilon^2 h_{[ab]}) + \text{c.c.} + O(\varepsilon^3) \quad [4.27]$$

The ε^2 appears in [4.27] for the same reason that it appears in [4.10]. This is a switch from what we might naively expect from arguments based on taking the $k \to 0$ limit to obtain the Einstein–Maxwell expressions,

[5]Note that Isaacson reverses our definitions of λ and ε.

where no ε^2 would appear. Substituting [4.27] into [4.23], the leading order in ε has the form:

$$[4.28] \qquad \Gamma'^{\alpha}{}_{[\beta\gamma]} = \exp\left(\frac{is}{\varepsilon}\right)\left(-\frac{i\varepsilon}{2}\right) g^{0\alpha\delta}(h_{[\beta\delta]}k_\gamma + h_{[\delta\gamma]}k_\beta + h_{[\beta\gamma]}k_\delta) + \text{c.c.} + \ldots$$

Since this is traceless

$$[4.29] \qquad g^{0\gamma\delta}k_\gamma h_{[\beta\delta]} = 0$$

After a little algebra, the leading term in $R'_{[\alpha\beta]}$ has the form:

$$[4.30] \qquad R'_{[\alpha\beta]} = \tfrac{1}{2}\exp(is/\varepsilon)\, g^{0\gamma\delta}k_\gamma k_\delta h_{[\alpha\beta]} + \text{c.c.} + O(\varepsilon)$$

Since $I'_{[\alpha\beta]}$ is of order ε^2, the field equation on the antisymmetric part of the curvature reads

$$[4.31] \qquad g^{0\gamma\delta}k_\gamma k_\delta h_{[\alpha\beta}k_{\rho]} = 0$$

which implies

$$[4.32] \qquad g^{0\alpha\beta}k_\alpha k_\beta = 0$$

or

$$[4.33] \qquad h_{[\alpha\beta}k_{\rho]} = 0$$

If [4.33] be true, then there exists v_α not proportional to k_α such that

$$[4.34] \qquad h_{[\alpha\beta]} = v_{[\alpha}k_{\beta]}$$

Substituting [4.34] into [4.29] in turn implies [4.32]. Hence, we find that k_α must be null with respect to the background metric.

Proceeding in a fashion analogous to that followed in the Einstein–Maxwell theory, we also find that the paths generated by k_α are geodesics of the background, i.e.,

$$[4.35] \qquad g^{0\alpha\beta}k_\alpha k_{\gamma\underset{0}{;}\beta} = 0$$

where the subscript 0 refers to a covariant derivative with respect to $\Gamma^{0\alpha}{}_{\beta\gamma}$.

In the above, we have obtained the first result which we need for the theory to be viable: to first order, electromagnetic and gravitational waves travel along null geodesics with respect to the background gravitational field. However, insofar as helping sort out which connection of the nonsymmetric field is preferred, we are not much further ahead, since, to this order, the connections are the same. To proceed further, we have two choices: either we could try to solve higher orders here (which would require detailed knowledged of the background) or leading order in the general background case. We will attempt the latter.

In the general background case we do not impose $g^0{}_{[\mu\nu]} = 0$. Dropping this condition complicates the field equations considerably. The zeroth-order equations involve g^0 and Γ^0 terms, and will not be discussed further here since they will provide no information on the behaviour of the high frequency disturbances. The first-order terms are:

$$[4.36] \qquad 0 = g'_{(\alpha\beta),\gamma} - g'_{(\alpha\delta)}\Gamma^{0\delta}{}_{(\beta\gamma)} - g^0{}_{(\alpha\delta)}\Gamma'^{\delta}{}_{(\beta\gamma)} - g'_{(\delta\beta)}\Gamma^{0\delta}{}_{(\alpha\gamma)} - g^0{}_{(\delta\beta)}\Gamma'^{\delta}{}_{(\alpha\gamma)} - g'_{[\alpha\delta]}\Gamma^{0\delta}{}_{[\gamma\beta]} - g^0{}_{[\alpha\delta]}\Gamma'^{\delta}{}_{[\gamma\beta]} - g'_{[\delta\beta]}\Gamma^{0\delta}{}_{[\alpha\gamma]} - g^0{}_{[\delta\beta]}\Gamma'^{\delta}{}_{[\alpha\gamma]}$$

$$[4.37] \qquad 0 = g'_{[\alpha\beta],\gamma} - g'_{[\alpha\delta]}\Gamma^{0\delta}{}_{(\beta\gamma)} - g^0{}_{[\alpha\delta]}\Gamma'^{\delta}{}_{(\beta\gamma)} - g'_{[\delta\beta]}\Gamma^{0\delta}{}_{(\alpha\gamma)} - g^0{}_{[\delta\beta]}\Gamma'^{\delta}{}_{(\alpha\gamma)} - g'_{(\alpha\delta)}\Gamma^{0\delta}{}_{[\gamma\beta]} - g^0{}_{(\alpha\delta)}\Gamma'^{\delta}{}_{[\gamma\beta]} - g'_{(\delta\beta)}\Gamma^{0\delta}{}_{[\alpha\gamma]} - g^0{}_{(\delta\beta)}\Gamma'^{\delta}{}_{[\alpha\gamma]}$$

$$[4.38] \qquad R'_{(\alpha\beta)} = \Gamma'^{\gamma}{}_{(\alpha\beta),\gamma} - \tfrac{1}{2}(\Gamma'^{\gamma}{}_{(\alpha\gamma),\beta} + \Gamma'^{\gamma}{}_{(\beta\gamma),\alpha}) - \Gamma^{0\gamma}{}_{(\alpha\delta)}\Gamma'^{\delta}{}_{(\gamma\beta)} - \Gamma'^{\gamma}{}_{(\alpha\delta)}\Gamma^{0\delta}{}_{(\gamma\beta)} + \Gamma^{0\gamma}{}_{[\alpha\delta]}\Gamma'^{\delta}{}_{[\gamma\beta]} + \Gamma'^{\gamma}{}_{[\alpha\delta]}\Gamma^{0\delta}{}_{[\gamma\beta]} + \Gamma^{0\gamma}{}_{(\alpha\beta)}\Gamma'^{\delta}{}_{(\gamma\delta)} + \Gamma'^{\gamma}{}_{(\alpha\beta)}\Gamma^{0\delta}{}_{(\gamma\delta)}$$

$$[4.39] \qquad R'_{[\alpha\beta]} = \Gamma'^{\gamma}{}_{[\alpha\beta],\gamma} - \Gamma^{0\gamma}{}_{(\alpha\delta)}\Gamma'^{\delta}{}_{[\gamma\beta]} - \Gamma'^{\gamma}{}_{(\alpha\delta)}\Gamma^{0\delta}{}_{[\gamma\beta]} - \Gamma^{0\gamma}{}_{[\alpha\delta]}\Gamma'^{\delta}{}_{(\gamma\beta)} - \Gamma'^{\gamma}{}_{[\alpha\delta]}\Gamma^{0\delta}{}_{(\gamma\beta)} + \Gamma^{0\gamma}{}_{[\alpha\beta]}\Gamma'^{\delta}{}_{(\gamma\delta)} + \Gamma'^{\gamma}{}_{[\alpha\beta]}\Gamma^{0\delta}{}_{(\gamma\delta)}$$

We observe in [4.36] and [4.37] that the symmetric and antisymmetric parts of the connection do not separate out cleanly. To simplify the equations somewhat, we put

$$[4.40] \qquad g'_{\alpha\beta} = \exp(is/\varepsilon)(-\varepsilon^2 h_{\alpha\beta}) + \text{c.c.} + O(\varepsilon^3)$$

and assume

$$\Gamma'^{\alpha}{}_{\mu\nu} = \exp(is/\varepsilon)(-i\varepsilon\Delta^{\alpha}{}_{\mu\nu}) + \text{c.c.} + O(\varepsilon^2) \quad [4.41]$$

so that [4.36] and [4.37] become

$$\Delta^{\gamma}{}_{(\alpha\beta)} = \tau^{0\gamma\rho}[\tfrac{1}{2}(h_{(\rho\beta)}k_{\alpha} + h_{(\alpha\rho)}k_{\beta} - h_{(\alpha\beta)}k_{\rho}) - g^{0}{}_{[\delta\beta]}\Delta^{\delta}{}_{[\rho\alpha]} - g^{0}{}_{[\alpha\delta]}\Delta^{\delta}{}_{[\beta\rho]}] \quad [4.42]$$

$$\Delta^{\gamma}{}_{[\alpha\beta]} = \tau^{0\gamma\rho}[\tfrac{1}{2}(h_{[\rho\beta]}k_{\alpha} + h_{[\alpha\rho]}k_{\beta} + h_{[\alpha\beta]}k_{\rho}) - g^{0}{}_{[\delta\beta]}\Delta^{\delta}{}_{(\rho\alpha)} - g^{0}{}_{[\alpha\delta]}\Delta^{\delta}{}_{(\beta\rho)}] \quad [4.43]$$

where

$$\tau^{0\alpha\rho}g^{0}{}_{(\rho\beta)} = \delta^{\alpha}{}_{\beta} \quad [4.44]$$

Unfortunately, it has not been possible to make any further progress towards solving these equations without invoking several assumptions. Since neither have we been able to justify these assumptions nor do they yield a clean solution, we will not consider them here. One comment worth making here is that decreasing the strength of the high frequency term in $g'_{(\mu\nu)}$ by adding another power of ε does not yield any simpler result. It should also be added that, unlike in general relativity, it no longer follows that k_{α} is geodesic if it is null and the gradient of a scalar.

Although we are not able to solve the case which would discriminate among the various connections, we can obtain a useful result if we relax our condition on the behaviour of $R_{\alpha\beta}$ in the high frequency limit. Isaacson (16) argues that ε^{-1} behaviour in the curvature is allowable in some circumstances. Following Isaacson we eliminate λ from the equations by replacing it with ε, so that the gravitational background case we considered before becomes

$$g_{[\alpha\beta]} = \varepsilon \exp(is/\varepsilon)\, h_{[\alpha\beta]} + \text{c.c.} + O(\varepsilon^2) \quad [4.45]$$

$$g_{(\alpha\beta)} = g^{0}{}_{(\alpha\beta)} + \varepsilon \exp(is/\varepsilon)\, h_{(\alpha\beta)} + \text{c.c.} + O(\varepsilon^2) \quad [4.46]$$

This is a slightly stronger result in that we now have a one parameter expansion whose convergence is easier to assess, at least in principle. Substituting

$$\Gamma^{\gamma}{}_{[\alpha\beta]} = \exp(is/\varepsilon)(i/2)g^{0(\gamma\rho)}(h_{[\rho\beta]}k_{\alpha} + h_{[\alpha\rho]}k_{\beta} + h_{[\alpha\beta]}k_{\rho}) + \text{c.c.} + O(\varepsilon) \quad [4.47]$$

into the curvature yields

$$R_{[\alpha\beta]} = -(1/\varepsilon)\exp(is/\varepsilon)\, g^{0(\gamma\rho)}h_{[\alpha\beta]}k_{\gamma}k_{\rho} + \text{c.c.} + O(1) \quad [4.48]$$

After some work, we are led to

$$g^{0(\alpha\beta)}k_{\alpha}k_{\beta} = 0 \quad [4.49]$$

In deriving the result, $g^{0(\alpha\beta)}$ appears, so that the electromagnetic perturbation moves on null geodesics of the connection compatible with the slowly varying background metric:

$$\Lambda^{0\gamma}{}_{(\alpha\beta)} = \tfrac{1}{2}g^{0(\gamma\rho)}(g^{0}{}_{(\rho\beta),\alpha} + g^{0}{}_{(\alpha\rho),\beta} - g^{0}{}_{(\alpha\beta),\rho}) \quad [4.50]$$

However, to this order the low frequency parts of $\Gamma^{\alpha}{}_{\beta\gamma}$ and $\Lambda^{\alpha}{}_{(\beta\gamma)}$ are the same, and so we cannot definitely state that $\Gamma^{\alpha}{}_{\beta\gamma}$ is ruled out.

5. Conclusions

The generalization of the nonsymmetric unified field theory which we have discussed in this paper has been shown in previous work to possess many desirable qualities not found in Einstein's original formulation. What we have tried to do here is look at the fundamental geometric and physical implications of the theory. We find that much of the structure found in the "non-unified" Einstein–Maxwell theory has been lost.

The first problem we face is the choice between which of three connections we should use to define the geodesics, or paths, of the theory. We find that one of the connections which appear in the field equations preserves the lengths of tangent vectors to the path under parallel propagation, but does not preserve the product of two arbitrary vectors, nor does it generate a path which is an extremal of arc length. The Levi-Civita connection which does generate a path that is an extremal of arc length does not arise in the field equations. However, it does allow for preserved products and for the construction of locally flat Lorentz frames, which cannot be obtained with either of the other connections.

To try to resolve which of these connections, if any, have physical meaning, we look to the propagation of light in the theory. First, we showed that indeed electromagnetic and gravitational plane waves do exist in a Minkowski background, a prerequisite. Next, we showed that, in the absence of a strong electromagnetic background, they propagate along the geodesics of the gravitational background (at this level of approximation, there is no difference between the connections). In the general background case, where we would be able to distinguish between connections, the problem remains unsolved.

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Appendix: Weak Field Approximation to Second Order

The curvature tensor in the weak field limit of Einstein's unified field theory has been derived previously (4, 20). We complete the calculations for our theory in this Appendix. In the weak field approximation, the metric tensor $g_{\mu\nu}$ is expanded in the series

[A1] $$g_{\mu\nu} = \eta_{\mu\nu} + \lambda g'_{\mu\nu} + \lambda^2 g''_{\mu\nu} + \ldots$$

where $\eta_{\mu\nu}$ is the flat space–time metric tensor and the $g'_{\mu\nu}$ are weak perturbing fields, the number of primes indicating the order of weakness. The elements of the contravariant tensor

[A2] $$g^{\mu\nu} = \eta^{\mu\nu} + \lambda g'^{\mu\nu} + \lambda^2 g''^{\mu\nu} + \ldots$$

can be found from the orthogonality condition:

[A3] $$g'^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\beta\nu}g'_{\beta\alpha}$$

[A4] $$g''^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\beta\nu}g''_{\beta\alpha} + \eta^{\mu\alpha}\eta^{\beta\nu}\eta^{\rho\sigma}g'_{\rho\alpha}g'_{\beta\sigma}$$

where we have carried the calculation to second order. We employ the same expansion for the connection Γ:

[A5] $$\Gamma^{\sigma}_{\mu\nu} = \lambda\Gamma'^{\sigma}_{\mu\nu} + \lambda^2\Gamma''^{\sigma}_{\mu\nu} + \ldots$$

Equation [1.5] can be used with [1.6], which now reads

[A6] $$\Gamma'^{\alpha}_{[\mu\alpha]} = 0, \quad \Gamma''^{\alpha}_{[\mu\alpha]} = 0$$

to obtain expressions for Γ' and Γ'':

[A7] $$\Gamma'^{\sigma}_{(\mu\nu)} = \tfrac{1}{2}\eta^{\sigma\alpha}(g'_{(\alpha\nu),\mu} + g'_{(\mu\alpha),\nu} - g'_{(\nu\mu),\alpha})$$

[A8] $$\Gamma''^{\sigma}_{(\mu\nu)} = \tfrac{1}{2}\eta^{\sigma\alpha}(g''_{(\alpha\nu),\mu} + g''_{(\mu\alpha),\nu} - g''_{(\nu\mu),\alpha} - 2g'_{[\beta\nu]}\Gamma'^{\beta}_{[\alpha\mu]} - 2g'_{[\mu\beta]}\Gamma'^{\beta}_{[\nu\alpha]} - 2g'_{(\alpha\beta)}\Gamma'^{\beta}_{(\mu\nu)})$$

[A9] $$\Gamma'^{\sigma}_{[\mu\nu]} = \tfrac{1}{2}\eta^{\sigma\alpha}(g'_{[\alpha\nu],\mu} + g'_{[\mu\alpha],\nu} - g'_{[\nu\mu],\alpha})$$

[A10] $$\Gamma''^{\sigma}_{[\mu\nu]} = \tfrac{1}{2}\eta^{\sigma\alpha}(g''_{[\alpha\nu],\mu} + g''_{[\mu\alpha],\nu} - g''_{[\nu\mu],\alpha} - 2g'_{[\beta\nu]}\Gamma'^{\beta}_{(\alpha\mu)} - 2g'_{[\mu\beta]}\Gamma'^{\beta}_{(\nu\alpha)} - 2g'_{(\alpha\beta)}\Gamma'^{\beta}_{[\mu\nu]})$$

This implies that the contracted curvature tensor has the form:

[A11] $$R'_{(\mu\nu)} = \Gamma'^{\alpha}_{(\mu\nu),\alpha} - \tfrac{1}{2}(\Gamma'^{\alpha}_{(\mu\alpha),\nu} + \Gamma'^{\alpha}_{(\nu\alpha),\mu})$$

[A12] $$R''_{(\mu\nu)} = \Gamma''^{\alpha}_{(\mu\nu),\alpha} - \tfrac{1}{2}(\Gamma''^{\alpha}_{(\mu\alpha),\nu} + \Gamma''^{\alpha}_{(\nu\alpha),\mu}) - \Gamma'^{\alpha}_{(\mu\beta)}\Gamma'^{\beta}_{(\alpha\nu)} - \Gamma'^{\alpha}_{[\mu\beta]}\Gamma'^{\beta}_{[\alpha\nu]} + \Gamma'^{\beta}_{(\alpha\beta)}\Gamma'^{\alpha}_{(\mu\nu)}$$

[A13] $$R'_{[\mu\nu]} = \Gamma'^{\alpha}_{[\mu\nu],\alpha}$$

[A14] $$R''_{[\mu\nu]} = \Gamma''^{\alpha}_{[\mu\nu],\alpha} - \Gamma'^{\alpha}_{(\mu\beta)}\Gamma'^{\beta}_{[\alpha\nu]} - \Gamma'^{\alpha}_{[\mu\beta]}\Gamma'^{\beta}_{(\alpha\nu)} + \Gamma'^{\beta}_{(\alpha\beta)}\Gamma'^{\alpha}_{[\mu\nu]}$$

Lastly, [1.10] yields

[A15] $$I'_{(\mu\nu)} = 0$$

[A16] $$I''_{(\mu\nu)} = (8\pi/k^2)(\eta^{\alpha\beta}g'_{[\beta\mu]}g'_{[\alpha\nu]} - \tfrac{1}{4}\eta_{\mu\nu}g'_{[\alpha\beta]}g'^{[\alpha\beta]})$$

[A17] $$I'_{[\mu\nu]} = -(8\pi/k^2)g'_{[\mu\nu]}$$

[A18] $$I''_{[\mu\nu]} = -(8\pi/k^2)g''_{[\mu\nu]}$$