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# Self-energy corrections to fermions in the presence of a thermal background

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The interaction of a fermion with a thermal background in the context of QED is investigated. Employing the matrix form of the real-time formalism for the propagators, the one-loop fermion self-energy is calculated for the general case of finite temperature and chemical potential. These results are shown to include others found in the recent literature as special cases. As an application, the shift of the electron's mass in a neutron star is calculated. It is shown that, at neutron-star densities, the shift is large compared to the free-particle electron mass, but small compared to the electron chemical potential.

## I. INTRODUCTION

Finite-temperature and -density (FTD) nonrelativistic field theories have been around for some time, but only in the last decade has there been interest in relativistic versions. This has been due largely to the discovery that broken gauge symmetries can be spontaneously restored at sufficiently high temperatures.[1,2] More recently, some workers have investigated other aspects of FTD field theories in an attempt to discover their effect on certain early-universe processes such as primordial nucleosynthesis.[3-5] The results, to date, have not altered the zero-temperature predictions significantly. Calculation of the one-loop QED mass shift for a point fermion in the presence of a background characterized by a nonzero ($T$) or chemical potential ($\mu$) (hereafter referred to as the thermal background) has already been attempted for several cases.[3,6-10] However, because particular assumptions have been made in some of these calculations, their results are not, in general, comparable.[11] One of our purposes here will be to do a general calculation which will allow a meaningful comparison of these special cases, and to confirm the domain of validity of each one. Further, since most calculations to date have been performed at zero chemical potential (of interest in the early universe, but not in neutron stars), our general calculation will include $\mu\neq 0$ effects.

The results of the general calculation of the mass shift at finite temperature and chemical potential are presented in the following section. To evaluate the general case, the real-time (matrix) propagator form of FTD field theory will be used (see Refs. 12–15). The results of our calculation are shown to contain the calculations of Refs. 3, 7, and 10 as special cases for $\mu=0$. Analytic results for $T=0$ special cases are also presented in an extension of Refs. 3, 7, and 10 using finite-chemical-potential propagators.[12-16]

As an application of the mass-shift calculation a simple model for a neutron star is presented in Sec. III to show the conditions to which one must go to give chemical potentials large enough to produce a noticeable mass shift. The results of all of the calculations are summarized in Sec. IV.

In the interest of clarity, the following notation will be used consistently throughout the paper: (i) quantities with a zero subscript (e.g., $\Sigma_0$) denote the $T=\mu=0$ contribution; (ii) quantities with a $\beta$ subscript (e.g., $\Sigma_\beta$) denote the FTD contribution; (iii) quantities with no subscript denote both the $T=\mu=0$ and FTD contributions (e.g., $\Sigma=\Sigma_0+\Sigma_\beta$).

## II. GENERAL CALCULATION OF THE MASS SHIFT

For a massive fermion, the bare propagator function (i.e., the 11 component of $S_0$, see Refs. 12–15 for details) is modified by the FTD self-energy correction shown in Fig. 1, to give

$$S^{11}\equiv S(K)=\frac{1}{\not K-\Sigma(K)+i\eta}\ ,$$

where

$$\Sigma(K)=-a\not K-b\not u+m_0-d \tag{2.1}$$

and $u$ is the four-velocity of the heat bath normalized by $u^\alpha u_\alpha=1$. The quantities $a$, $b$, and $d$ are Lorentz-invariant functions which can depend on two Lorentz scalars

$$\omega=K^\alpha u_\alpha \tag{2.2}$$

and

$$k=[(K^\alpha u_\alpha)^2-K^2]^{1/2} \tag{2.3}$$

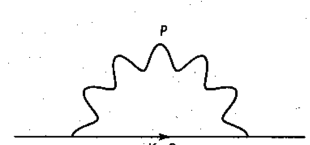


FIG. 1. One-loop contribution to the fermion self-energy illustrating the momentum labels used in the text.

 

such that

$$K^2=\omega^2-k^2 . \tag{2.4}$$

In general, $a$, $b$, and $d$ are complex, but we shall ignore the imaginary parts in all our calculations. Justification for doing this is briefly outlined at the end of this section.

The fermion self-energy is

$$\Sigma(K)=ie^2\int\frac{d^4P}{(2\pi)^4}\gamma^\mu D_{\mu\nu}(P)S(K-P)\gamma^\nu , \tag{2.5}$$

where

$$D_{\mu\nu}(P)=-g_{\mu\nu}\left[\frac{1}{P^2+i\eta}-i\Gamma_b(P)\right],$$

$$S(P)=(\not{P}+m_0)\left[\frac{1}{P^2-{m_0}^2+i\eta}+i\Gamma_f(P)\right],$$

and

$$\Gamma_b(P)=2\pi\delta(P^2)n_b(P) ,$$

$$\Gamma_f(P)=2\pi\delta(P^2-{m_0}^2)[\theta(P_0)n_f^+(P)+\theta(-P_0)n_f^-(P)] .$$

The $n$'s are the distribution functions

$$n_b(P)=[\exp(\beta|P\cdot u|)-1]^{-1} ,$$

$$n_f^\pm(P)=\{\exp[\beta(|P\cdot u|\pm\mu)]+1\}^{-1}$$

with $\beta$ the inverse temperature, and $\mu<0$ implying the particle and the background have the same sign for net charge. Ignoring the $T=\mu=0$ piece in Eq. (2.5) leaves

$$\Sigma_\beta(K)=-ie^2\int\frac{d^4P}{(2\pi)^4}\gamma_\nu(\not{K}-\not{P})\gamma^\nu\left[\frac{i\Gamma_f(K-P)}{P^2}-\frac{i\Gamma_b(P)}{(K-P)^2}-\Gamma_b(P)\Gamma_f(K-P)\right] . \tag{2.6}$$

For the remainder of this section, we will concern ourselves with only the first two terms of Eq. (2.6), i.e., the real part of $\Sigma_\beta(K)$. The full fermion propagator will exhibit a pole when

$$(1+a)^2(\omega^2-k^2)+2(1+a)b\omega+b^2-c^2=0 , \tag{2.7}$$

where $c=m_0-d$.

Writing Eq. (2.7) in the form of a dispersion relation gives

$$\omega=\frac{-b\pm[c^2+(1+a)^2k^2]^{1/2}}{1+a} . \tag{2.8}$$

This expression is rather complicated—especially when one remembers that $a$, $b$, and $c$ are, of necessity, functions of $\omega$ and $k$. Rather than attempt to solve Eq. (2.8) for the complete dispersion relation, we will content ourselves with calculating the FTD mass shift which we define by

$$\delta m\equiv\lim_{k\to0}\omega-m_0 . \tag{2.9}$$

Applying this definition to Eq. (2.8) we find

$$\delta m=\lim_{k\to0}[\tfrac{1}{4}\mathrm{Tr}(\not{u}\,\mathrm{Re}\Sigma_\beta)+\tfrac{1}{4}\mathrm{Tr}(\mathrm{Re}\Sigma_\beta)] . \tag{2.10}$$

It can be shown that

$$\mathrm{Re}\Sigma_\beta(K)=\frac{\alpha}{\pi^2}\int d^4P\left[\frac{\not{P}+2m_0}{(P+K)^2}\delta(P^2-{m_0}^2)[\theta(-P_0)n_f^+(K-P)+\theta(P_0)n_f^-(K-P)]+\frac{\not{K}-\not{P}-2m_0}{(K+P)^2-{m_0}^2}\delta(P^2)n_b(P)\right] , \tag{2.11}$$

where $\alpha=e^2/4\pi$.

A change of variables, $P\to P+K$ for the first term and $P\to-P$ for the second term, has been made. $\frac{1}{4}\mathrm{Tr}(\not{u}\,\mathrm{Re}\Sigma_\beta)$ and $\frac{1}{4}\mathrm{Tr}(\mathrm{Re}\Sigma_\beta)$ can now be evaluated. We find

$$\tfrac{1}{4}\mathrm{Tr}(\not{u}\,\mathrm{Re}\Sigma_\beta)=\frac{\alpha}{2\pi k}\int_0^\infty dp\{[\omega(L_1^++L_1^-)+p(L_1^+-L_1^-)]n_b(p)+p[L_2^+n_f^-(r)-L_2^-n_f^+(r)]\} , \tag{2.12}$$

$$\tfrac{1}{4}\mathrm{Tr}(\mathrm{Re}\Sigma_\beta)=\frac{\alpha m_0}{\pi k}\int_0^\infty dp\left[\frac{p}{r}[L_2^+n_f^-(r)+L_2^-n_f^+(r)]-[L_1^++L_1^-]n_b(p)\right] , \tag{2.13}$$

where

$$L_1^{\pm}=\pm\ln\left[\frac{p(\omega+k)\pm(K^2-m_0{}^2)/2}{p(\omega-k)\pm(K^2-m_0{}^2)/2}\right],$$

$$L_2^{\pm}=\pm\ln\left[\frac{r\omega+pk\pm(K^2+m_0{}^2)/2}{r\omega-pk\pm(K^2+m_0{}^2)/2}\right],$$

$$p=|\mathbf{P}|,\quad r=(p^2+m_0{}^2)^{1/2}. \tag{2.14}$$

We now take the $k\to 0$ limit and find

$$\frac{\delta m}{m_0}=I_B+I_{F^+}+I_{F^-}; \tag{2.15}$$

where

$$I_B=\frac{4\alpha w}{\pi}\int_0^\infty dx\,x\frac{1+2x^2+2(w-1/w)-w^2}{4x^2w^2-(w^2-1)^2}\frac{1}{e^{x/t}-1}, \tag{2.16}$$

$$I_{F^+}=\frac{2\alpha}{\pi}\int_0^\infty dx\frac{x^2}{y}\frac{y-2}{2yw-w^2-1}\frac{1}{e^{(y+v)/t}+1}, \tag{2.17}$$

$$I_{F^-}=\frac{2\alpha}{\pi}\int_0^\infty dx\frac{x^2}{y}\frac{y+2}{2yw+w^2+1}\frac{1}{e^{(y-v)/t}+1}, \tag{2.18}$$

and

$$w=\frac{\omega}{m_0},\quad t=\frac{1}{\beta m_0},\quad v=\frac{\mu}{m_0},\quad y=(x^2+1)^{1/2}.$$

Equations (2.16)–(2.18) yield an implicit equation for $m$ which cannot be solved analytically. We have solved these equations numerically, an example of the solution to Eq. (2.15) for various values of $\mu$ being shown in Fig. 2. The small-$T$ region is shown in more detail in Fig. 3, and it can be seen that for $\mu/m_0\leq -1$ it is possible to have a negative mass shift, although its magnitude is small. In Fig. 4 the results for $\mu=0$ are shown along with the results of a similar calculation performed in Refs. 3 and 7. As one would expect the general solution agrees with the results found in Ref. 3 at $T\ll m_0$, and with Ref. 7 at $T\gg m_0$.

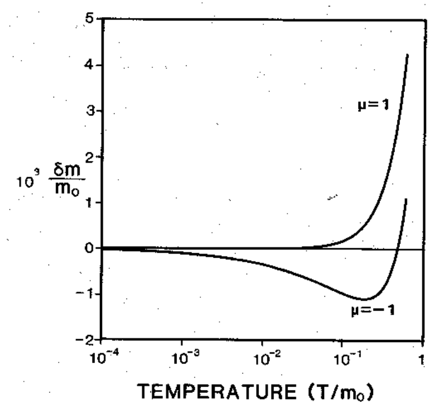


FIG. 3. An enlargement of the low-$T$ portion of Fig. 2 showing region of negative mass shift ($\mu$ is shown in units of $m_0$).

Before leaving the general calculation and moving on to discuss some special cases we would like to address an issue raised at the beginning of this section. It was pointed out that the coefficients $a$, $b$, and $d$ are, in general, complex. Including them in Eq. (2.1) would lead to a condition for poles similar to Eq. (2.7) except that terms like $a^2$ would be replaced by $(\text{Re}a)^2-(\text{Im}a)^2$. In order to justify ignoring $\text{Im}a^2$ we should show

$$\left[\frac{\text{Im}a}{\text{Re}a}\right]^2\ll 1.$$

We have not succeeded in demonstrating this for the general case; however, we have studied several special cases with the following result. For massless fermions, in the $k\to 0$ limit, we find

$$\left[\frac{\text{Im}a}{\text{Re}a}\right]^2=\frac{\beta^2\omega^2}{64\pi^2}. \tag{2.19}$$

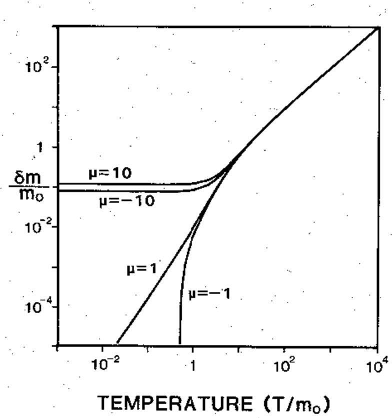


FIG. 2. Numerical solution of Eq. (2.15) showing the FTD mass shift as a function of $T$ for various values of $\mu$ ($\mu$ is shown in units of $m_0$).

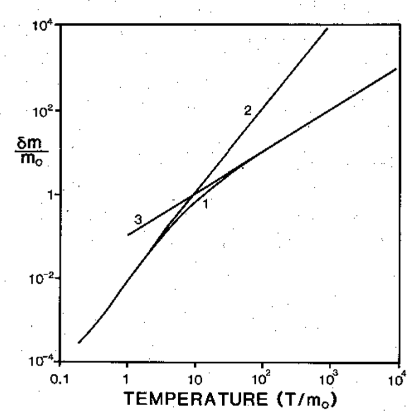


FIG. 4. Comparison of mass-shift calculations performed: (1) in this paper; (2) in Ref. 3; (3) in Ref. 7.

Unfortunately, the ratio depends on $\omega$, and of course the value of $\omega$ depends on whether or not terms like Im$a$ are ignored. We can, however, do a consistency check. Ignoring Im$a$ leads to the result $\omega^2=(\alpha\pi/2)T$ so the ratio in Eq. (2.19) becomes

$$\frac{\alpha}{132\pi} \ll 1 \; .$$

Put another way, inclusion of terms like Im$a$ would be necessary only if the resulting correction factor was on the order of $8\pi/\alpha \sim 3000$. Calculations for other special cases give similar results.

### A. Special cases

One can obtain expressions for the mass shift at asymptotic values of small and large $\mu/m_0$ by using the same methods as were employed in Refs. 3, 7, and 10 for small and large $T/m$. For example, in Ref. 3 a low-temperature approximation was made by neglecting the heat-bath term and evaluating the self-energy in the $k=0$ limit. Performing the same calculation for $T=0$, $\mu\neq 0$, we find

$$\mathrm{Re}\Sigma_\beta=\frac{\alpha m_0}{2\pi}\{(v^2-1)^{1/2}(v\pm 2)-3\ln[(v^2-1)^{1/2}+v]\} \; , \tag{2.20}$$

where $v$ is defined below Eq. (2.11), and the upper (lower) sign in Eq. (2.20) refers to $\mu>0$ ($\mu<0$). Comparison of this approximate expression with the results of numerically integrating Eqs. (2.16)–(2.18) shows reasonable agreement ($\lesssim 10\%$ error) for the range

$$-10 \lesssim \frac{\mu}{m_0} \lesssim 10 \; . \tag{2.21}$$

To obtain analytic expressions for $\mu/m_0 \gg 1$, the techniques used in Ref. 7 for massless fermions can easily be extended to $\mu\neq 0$. After some algebra, one finds that

$$M^2=\frac{\alpha}{2\pi}(\mu^2+\pi^2T^2) \; , \tag{2.22}$$

where $M^2$ is defined by

$$M^2\equiv\tfrac{1}{2}\mathrm{Tr}(K\,\mathrm{Re}\Sigma_\beta) \; . \tag{2.23}$$

Equation (2.22) contains both the $\mu=0$ results of Refs. 7 and 10, as well as the $T=0$ result from Ref. 10 as special cases.

It can be demonstrated that $M$ in Eq. (2.22) actually does enter into the dispersion relation between $\omega$ and $k$ like a mass. Working in dimensionless variables $w\equiv\omega/M$ and $\kappa=k/M$, the dispersion relation for massless fermions reads

$$w-\kappa=\frac{1}{\kappa}\left[1+\left[1+\frac{w}{\kappa}\right]\tfrac{1}{2}\ln\left[\frac{w_+}{w_-}\right]\right] , \tag{2.24}$$

where $w_\pm=(w\pm\kappa)/2$. A plot of $(w^2-\kappa^2)^{1/2}$ generated by Eq. (2.24) is shown in Fig. 5(a). Of course, for a free massive particle one would expect $w^2-\kappa^2=1$, so that one can see that deviations of up to 40% from the $k=0$ value are found for the effective mass as a function of $k$. On the other hand, one would expect $w$ and $w_f\equiv(\kappa^2+1)^{1/2}$ to converge as $\kappa\to\infty$, and this is shown in Fig. 5(b). Hence, the greatest effect on the behavior of the free massless electron's energy caused by introducing it into the medium at finite $T,\mu$ will be at low values of $k$.

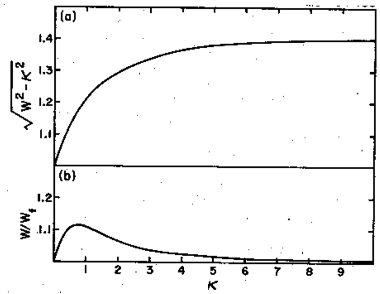


FIG. 5. (a) Effective mass (in units of $M$) from $w^2-\kappa^2$ shown as a function of $\kappa$. (b) Ratio of $w$ calculated from Eq. (2.16) to $w_f=(\kappa^2+1)^{1/2}$, again as function of $\kappa$.

## III. NEUTRON STARS

Because the electromagnetic mass shift at finite chemical potential goes like $\sqrt{\alpha/2\pi}\mu$, its physical effects are going to be small. To observe such effects, one must either find a physical observable which can be measured very accurately (for example, the magnetic moment) such that a small value of $\mu$ could yield a detectable result, or find an environment with a very large value for $\mu$. One such possibility would be a neutron star.[17] To determine whether the chemical potentials are large enough to produce an observable effect, we adopt a simplified model of a neutron star and calculate the appropriate electron chemical potential and mass shift. Construction of a detailed model to calculate the changes in neutron-star properties arising from the electron mass shift is beyond the scope of this paper.

For the model, a uniform gas of electrons, protons, and neutrons will be chosen for the neutron-star matter (clearly, such a model will not be valid at high densities where strong-interaction effects will become important). Chemical potentials $\mu_e$, $\mu_p$, and $\mu_n$, which include the mass, are assigned, respectively, to the electrons, protons, and neutrons present with number densities $n_e$, $n_p$, and $n_n$. For a Fermi gas with a Fermi energy much larger than the temperature, these quantities can be related via $\mu^2=m^2+(3\pi^2 n)^{2/3}$. If $\mu_e+\mu_p$ exceeds $\mu_n$, then electrons will be captured by protons until the number densities change such that $\mu_e+\mu_p=\mu_n$. Hence, for each value of $n_n$, there will be a value of $\mu_e$ which satisfies the $\beta$-stability condition.

Assuming local electrical neutrality so that $n_e=n_p$, the chemical potential equality then yields

$$\mu_e+(\mu_e{}^2+m_p{}^2-m_e{}^2)^{1/2}=[(3\pi^2n_n)^{2/3}+m_n{}^2]^{1/2} \; . \tag{3.1}$$

The solution of this equation for $\mu_e$ as a function of $n_n$ is shown in Fig. 6. At small $n_n$, $\mu_e\sim m_n-m_p$, while for large $n_n$ (but not so large that the neutrons are relativistic)

$$\mu_e\approx(3\pi^2n_n)^{2/3}/2m_n \; . \tag{3.2}$$

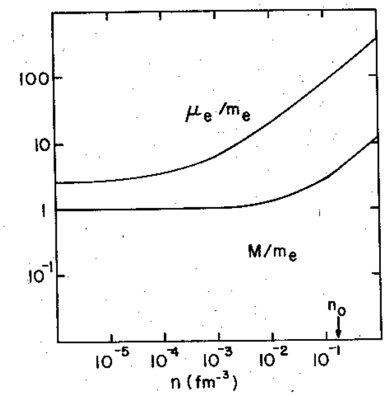


FIG. 6. Expected value of the electron chemical potential and electron mass shown as a function of neutron number density. [Normal nuclear matter density ($n_0$) is indicated for comparison.]

Both of these limits are obvious from Fig. 6. This figure includes a region in which $\mu_e$ exceeds $m_\pi$, although obviously $e^- \to \pi^- + \nu_e$ would be allowed in this region (see Ref. 18).

From Fig. 6 it can be seen that over much of the neutron density range of interest in the formation of a neutron star, $\mu_e$ is large compared to $m_e$. It is found that the finite $\mu$ electron mass ($\equiv M$) from Eq. (2.15) becomes substantial compared to $m_e$, as is shown in Fig. 6, but small compared to $\mu_e$ ($M/\mu_e \sim 1/30$). The larger electron mass will lead to an earlier onset of electron capture in the formation of the neutron star, and hence an increase in the rate of neutrino emission. Since the neutrino mass is changed only by the weak interaction, its mass shift would be very small indeed and would not compensate for the increased electron mass. Similarly, the abundance of electrons in the neutron-star core would be lowered.

## IV. SUMMARY

We have performed a calculation of the QED self-energy corrections for a massive fermion at finite $T$ and $\mu$. It is found that the solution contains for zero chemical potential both a high- and low-temperature limit which agrees with two previous special-case calculations. We have also extended the calculation to finite chemical potential and shown that the magnitude of the corrections is similar to that found for $\mu=0$, $T\neq 0$. Analytic expressions for large and small $\mu$ at $T=0$ are also given.

In an attempt to find a system which has a large enough chemical potential such that these finite $T,\mu$ effects might be observed, the interior of a neutron star was considered. In a simplified model, it was shown that the electron mass shift in a neutron star may be several times its rest mass, and hence the rate of cooling of the star will be changed. However, the mass shift is still small (3.5%) compared to the chemical potential, so one does not expect there to be large changes to any measurable quantities. Other evidence for this conclusion can be found in a calculation of the energy density of a gas of electrons, positrons, and photons at finite chemical potential, where it has been shown[19] that the ratio of the corrections to the energy density to the ideal (free particle) value is of order $\alpha/2\pi$.

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