Two-particle correlation functions in the thermal model and nuclear interferometry descriptions

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The equivalence of the thermal model and the conventional zero lifetime Hanbury-Brown/Twiss descriptions of the two-particle momentum-space correlation function is demonstrated. A simplified expression for the correlation function is presented and tested against recent composite particle correlation measurements. The expression allows a more direct analysis of the measurements and predicts novel behavior for certain interaction potentials.

The technique of nuclear interferometry involving correlations among pions,[1–4] protons,[5–9] and light nuclei[10–12] has been successful in helping to map out the space-time trajectory of a heavy ion reaction. Recently,[13] $d\alpha$ coincidence measurements were used to reconstruct a ^{6}Li excitation spectrum (not a correlation function) whose form was consistent with ^{6}Li excited state emission from a thermalized source with a temperature of about 5 MeV. Is this thermal model description of the excited state yields in conflict with the conventional nuclear interferometry description?

A system in thermal equilibrium may have multiparticle correlations for a variety of reasons, just as in the nuclear interferometry approach. Indeed, in the zero lifetime limit of the interferometry approach, one would expect the correlations to be the same for a uniform source as in the thermal model. To demonstrate this, we choose a uniform source of radius R_u and calculate the change in the correlation function in both models arising from the nuclear interaction term.

In the thermal model, the change in the density of states Δg_l arising from the interaction term for a particular partial wave l with phase shift δ_l is[14] (ignoring spin)

$$\Delta g_l = \frac{2l+1}{\pi}\frac{d\delta_l(p)}{dp} , \tag{1}$$

where p is the relative momentum of the two particle system. Compared to the free particle density of states, this implies a contribution to the correlation function $\Delta R(p)$ of

$$\Delta R(p) = \frac{3}{2}\frac{1}{R_u^3 k^2}(2l+1)\frac{d\delta_l(p)}{dk} , \tag{2}$$

where k is the wave vector corresponding to p.

In the conventional nuclear interferometry approach the correlation function for a uniform source function is given by,

$$\Delta R(p) = \frac{3}{4\pi R_u^3}\int_0^{2R_u}[\,|\psi(r,p)|^2-1]\times\left[1-\frac{3}{4}\left[\frac{r}{R_u}\right]+\frac{1}{16}\left[\frac{r}{R_u}\right]^3\right]d^3r , \tag{3}$$

where ψ^2 is the (summed, spin weighted) relative wave function of the two particles (see Refs. 1, 5, 10, and 11 for specific examples). Although the individual source terms for particles 1 and 2 are uniform in space, after integrating over the center of mass coordinate, the distribution in the relative coordinate r is not uniform; thus the second two terms in the large brackets in Eq. (3) have R_u in the denominator. For sufficiently large R_u these two terms can be neglected since the expression in the first set of brackets guarantees that the major contribution to the integral comes from small r. Assuming that at the upper limit the wave function may be replaced by its asymptotic form we have[15] (see the Appendix for details),

$$\frac{3}{R_u^3}\int_0^{2R_u}|\psi|^2 r^2 dr = \frac{3}{2R_u^3 k^2}\sum_l (2l+1)\left[-\frac{1}{2k}\sin(4kR_u+2\delta_l)+2R_u+\frac{d\delta_l}{dk}\right] . \tag{4}$$

The R_u dependence in the brackets in Eq. (4) is largely cancelled by the means of a similar expansion,

$$\frac{3}{R_u^3}\int_0^{2R_u} r^2 dr = \frac{3}{R_u^3}\sum_l (2l+1)\int_0^{2R_u}[rj_l(kr)]^2 dr . \tag{5}$$

Extracting the nuclear part, one recovers Eq. (2) as required.

This expression for the correlation function will be most valid in the region of the resonant states, where $d\delta_l/dk$ is large and may dominate over Coulomb and other contributions. To test its accuracy, we use Eqs. (2) and (3) to predict the peak in the $d\alpha$ correlation function at the resonant momentum $\Delta p = 42$ MeV/c. (Another resonance at 83 MeV/c is less useful since $d\delta_l/dk$ is smaller.) With the phase shifts associated with the potential model analysis of Ref. 11, Eq. (2) yields $R = 93$ and 20 for $R_u = 6$ and 10 fm, respectively. We can compare this with a numerical integration of Eq. (3). Taking the leading term [as was done in deriving Eq. (4)], ΔR is calculated to be 134 and 29 for $R_u = 6$ and 10 fm, respectively. However, when finite size effects are included in a full evaluation of Eq. (3), the predicted correlation function changes to 71 and 20, respectively (where the Coulomb term has been included). Thus we find that finite size effects are important and that Eq. (2) is not particularly accurate, with about 30% error for typical source radii. One would expect similar errors due to finite size effects in the thermal model. In any event we see that the thermal and the zero lifetime limit of the interferometry approach do give similar results and become the same for large source sizes.

Equation (2) can be easily extended to more general source distributions than the uniform distribution. In fact it turns out to be much more accurate for smoothly varying distributions such as Gaussians. Consider a source function $g(r)$ for each particle. We define the two-particle relative source function $f(r)$ by integrating over the center of mass coordinate; thus

$$f(r) = \int d^3R\, g\left[\mathbf{R} + \frac{m_1}{m_1+m_2}\mathbf{r}\right] g\left[\mathbf{R} - \frac{m_2}{m_1+m_2}\mathbf{r}\right]. \quad (6)$$

Assuming that $f(r)$ is slowly varying near $r=0$, we can use techniques similar to the uniform case (the details are again relegated to the Appendix). The result is

$$\Delta R(p) = \frac{f(0)}{\int d^3r\, f(r)} \frac{2\pi}{p^2} \sum_l (2l+1) \frac{d\delta_l(p)}{dp}. \quad (7)$$

In the limit of a uniform distribution for $g(r)$ the factor $\lambda = f(0)/\int d^3r\, f(r)$ becomes $1/V$, where V is the volume, and Eq. (7) reduces to Eq. (2); for a Gaussian source, $g(r) = \exp(-r^2/r_0^2)$, λ becomes $1/(\sqrt{2\pi}r_0)^3$.

As an application, we consider a Gaussian source. Then, for the $d\alpha$ peak discussed previously, the correlation function is predicted to have values of 74 and 28 for $r=4$ and 6 fm, respectively. The approximate form Eq. (7) gives 83 and 25, in good agreement with the exact results.

Returning to Eq. (2), there are several points about ΔR worth making. The first is that it is independent of the temperature, even in thermal models. Thus, the excitation function and the correlation function provide complementary information. The ratio of the peak areas in the excitation function depends only on the temperature of the system, at least as a first approximation. In contrast the ratio of the peaks in the correlation function depends mainly on the source size and to a first approximation is independent of the temperature. The second is that even though the interaction between the two particles is attractive, there may be a suppression rather than an enhancement in the correlation function. When there is one bound state, the phase shift starts from π at low energies and may either increase or decrease depending on the strength of the potential. If the phase shift decreases its derivative is negative: the same as for a repulsive potential.

As an example consider pd correlations. Here the bound state is the ^{3}He nucleus and the phase shift does decrease. The calculations were carried out as in Ref. 11. The low energy phase shifts[16] were fitted with a Woods-Saxon potential, with the potential parameters given in Table I. A Gaussian source with zero lifetime was used. Two values of r_0 were used, 4 and 8 fm, and Fig. 1 shows the predicted correlation with and without the nuclear terms. The shift is substantial and should be experimentally detectable.

Aside from the physics applications of Eq. (7), it is also useful simply for allowing rapid estimation of the correlation function. We have shown its accuracy above for the $d\alpha$ correlation function; we now look at some nonidentical elementary particle combinations. Just as consideration of nonidentical composite particle correlations (pd, pα, and $d\alpha$) has provided much reaction mechanism information at low energies, πp, Kπ, and Kp pairs may provide extra information at higher energies.

Taking the π^+p and $K^+\pi^+$ combinations as examples, the derivatives of the phase shifts[17] in the $p_{\text{c.m.}} = 100$ MeV region are -0.2 (s wave) and $+0.4$ (p wave) rad fm for π^+p and -0.22 rad fm (s wave) for $K^+\pi^+$. Using the Gaussian source parametrization, the predicted change in the correlation function for $r_0 = 4$ fm is 0.02 for π^+p and -0.005 for $K^+\pi^+$ at $p = 100$ MeV. For smaller Δp the shifts will be larger. Consequently we see that these pairs will probably be more sensitive to the Coulomb interaction than the nuclear one.

To summarize, we have derived a simplified expression for the conventional nuclear interferometry description of the two-particle correlation function. Numerical tests have shown that it is a reasonable approximation for a Gaussian source, less reasonable for a uniform source of small dimension. For large uniform sources, the expression is identical to what is found in the thermal model. The expression can also be used to demonstrate that the enhancement or suppression of the correlation function depends on the sign of the derivative of the phase shift, rather than on the sign of the interaction potential. The

TABLE I. Potential parameters from pd phase shifts assuming a Woods-Saxon form for the potential.

S	L	V_0 (MeV)	R_{WS} (fm)	a_{WS} (fm)
$\frac{1}{2}$	0	-29.754	2.826	1.187
$\frac{1}{2}$	1	-8.214	2.962	0.259
$\frac{1}{2}$	2	-7.849	2.974	0.991
$\frac{3}{2}$	0	-18.115	2.837	0.9655
$\frac{3}{2}$	1	-13.10	2.067	1.578
$\frac{3}{2}$	2	$+14.878$	2.527	1.235

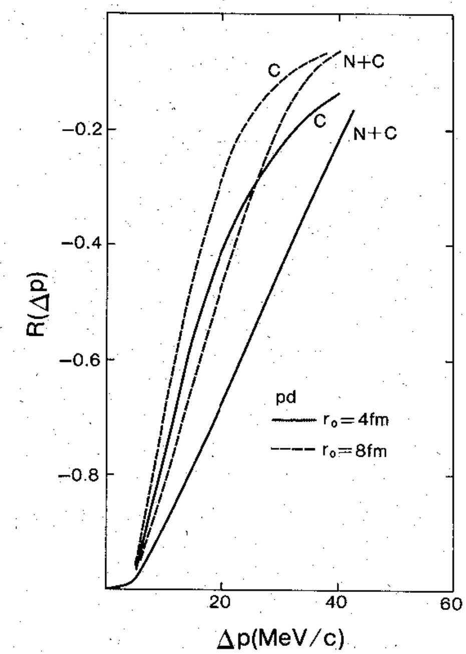

FIG. 1. Predictions for pd correlation function with (NC) and without (C) the nuclear term. The curves are shown with two values of r_0, 4 and 8 fm.

pd correlation function is calculated as an example of this effect. Finally, the usefulness of the approximation in estimating the magnitude of the nuclear contributions to the correlation function is illustrated for πp and Kπ pairs.

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APPENDIX

In this appendix the expressions for the two-body correlation function are derived. We assume a single particle source function $g(r)$. The correlation function can then be written

$$\Delta R = \frac{\int d^3r_1 d^3r_2 [\,|\psi(r_1,r_2)|^2 - 1]g(r_1)g(r_2)}{\left[\int d^3r\, g(r)\right]^2}, \quad \text{(A1)}$$

where $\psi(r_1,r_2)$ is the two-body wave function. The two-body interaction only affects the relative coordinate so $\psi(r_1,r_2)$ is just a plane wave in the center of mass coordinate, $\mathbf{R}=(\mathbf{r}_1+\mathbf{r}_2)/2$, and $|\psi(r_1,r_2)|$ depends only on the relative coordinate $\mathbf{r}=(\mathbf{r}_1-\mathbf{r}_2)$. Defining a relative function

$$f(r) = \int d^3R\, g\left[\mathbf{R}+\frac{m_1}{m_1+m_2}\mathbf{r}\right] g\left[\mathbf{R}-\frac{m_2}{m_1+m_2}\mathbf{r}\right] \quad \text{(A2)}$$

Eq. (A1) becomes

$$\Delta R = \frac{\int d^3r [\,|\psi(r)|^2 - 1]f(r)}{\int d^3r\, f(r)} . \quad \text{(A3)}$$

We now expand $\psi(r)$ and 1 in partial waves and assume $f(r)$ is spherically symmetric. The result is

$$\Delta R = \frac{4\pi \sum_l (2l+1)}{\int d^3r\, f(r)} \int dr\, r^2 [\phi_l^2(r) - j_l^2(kr)] f(r) , \quad \text{(A4)}$$

where $\phi_l(r)$ is the relative wave function in the lth partial wave and $j_l(kr)$ a spherical Bessel function. For simplicity we have ignored spin. We now concentrate explicitly on the integral

$$I = \int dr\, r^2 [\phi_l^2(r) - j_l^2(kr)] f(r) . \quad \text{(A5)}$$

Asymptotically $\phi_l(r)$ goes like

$$\sin(kr - l\pi/2 + \delta_l)/kr$$

while $j_l(kr)$ goes like

$$\sin(kr - l\pi/2)/kr .$$

We now write

$$\begin{aligned} I = \int_0^\infty dr\, r^2 [&\phi_l^2(r) - \sin^2(kr - l\pi/2+\delta_l)/(kr)^2 - j_l^2(kr) \\ &+ \sin^2(kr - l\pi/2)/(kr)^2] f(r) \\ + \int_0^\infty \frac{dr}{k^2} [&\sin^2(kr - l\pi/2+\delta_l) \\ &- \sin^2(kr - l\pi/2)] f(r) . \end{aligned} \quad \text{(A6)}$$

For the first integral we may restrict the upper limit since the quantity in square brackets goes to zero. If $f(r)$ is sufficiently smooth we may replace it by $f(0)$ inside this integral. Thus we have

$$\begin{aligned} I = f(0) \int_0^{R_0} dr\, r^2 [&\phi_l^2(r) - \sin^2(kr - l\pi/2+\delta_l)/(kr)^2 \\ &- j_l^2(kr) + \sin^2(kr - l\pi/2)/(kr)^2] \\ + \int_0^\infty \frac{dr}{k^2} &\sin\delta_l \sin(2kr - l\pi + \delta_l) f(r) , \end{aligned} \quad \text{(A7)}$$

where R_0 (not the center of mass position) is assumed to be sufficiently large that $\phi_l(r)$ and $j_l(kr)$ have reached their asymptotic form and sufficiently small that $f(r)$ can be assumed to be constant. The second integral was simplified using trigonometric identities. For the first term we follow a procedure very similar to that used in Ref. 15 for doing the effective range expansion. We define $u(r,k)=\phi_l(r)kr$, then $u_l(r)$ satisfies the equation

$$\frac{d^2}{dr^2}u_l(r,k) - \left[2\mu V(r) + \frac{l(l+1)}{r^2}\right] u_l(r,k) = -k^2 u_l(r,k) . \quad \text{(A8)}$$

The same equation (with k replaced by k') is satisfied by $u_l(r,k')$. Multiplying the first of these equations by $u_l(r,k')$, the second by $u_l(r,k)$, and taking the difference we have after an integration by parts

$$\left[u_l(r,k')\frac{d}{dr}u_l(r,k)-u_l(r,k)\frac{d}{dr}u_l(r,k')\right]_{r_a}^{r_b}=(k'^2-k^2)\int_{r_a}^{r_b}dr\,u_l(r,k)u_l(r,k') . \quad (A9)$$

We now let k' go to k and combine Eq. (A9) with similar results for the other parts of the first integral in Eq. (A7). Letting r_a go to zero and r_b to R the result is

$$I=\frac{f(0)}{2k^2}\frac{d\delta_l}{dk}-\frac{f(0)}{2k^3}\sin\delta_l\cos(-l\pi-\delta_l)+\frac{\sin\delta_l}{k^2}\int_0^\infty \sin(2kr-l\pi+\delta_l)f(r)dr . \quad (A10)$$

Note that this equation is independent of R. This can be rewritten as

$$I=\frac{f(0)}{2k^2}\frac{d\delta_l}{dk}+\frac{\sin\delta_l}{2k^3}\int_0^\infty \cos(2kr-l\pi+\delta_l)\frac{df(r)}{dr}dr . \quad (A11)$$

The second term can be expanded by successive integration by parts into an asymptotic series

$$I=\frac{f(0)}{2k^2}\frac{d\delta_l}{dk}-\frac{\sin\delta_l}{4k^4}\sin(-l\pi+\delta_l)\frac{df(0)}{dr}-\frac{\sin\delta_l}{8k^4}\cos(-l\pi+\delta_l)\frac{d^2f(0)}{dr^2}+\cdots . \quad (A12)$$

If $g(r)=\theta(r_0-r)$ then $df(0)/dr$ goes like $\sim 1/r_0$ while if $g(r)$ is smooth $df(0)/dr$ is zero and $d^2f(0)/dr^2$ goes like $1/a^2$, where a is a length scale related to the size of the source. This indicates why the simple formula works better for the Gaussian than the uniform source. In any event in the limit of a large source we have

$$\Delta R=\frac{f(0)}{\int d^3r\,f(r)}\frac{2\pi}{k^2}\sum_l(2l+1)\frac{d\delta_l}{dk} . \quad (A13)$$

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