Neoclassical Growth:  A First Model

Economics 105
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Consider a model of the economy in which output  (income) is a function of labour, L, and capital, K:   Y= F(K,L).  Written in per capita terms, where per capita output (Y/L), is equal to **y**, we have:  $y=f(K/L)=f(k)$, where **k**=(K/L), the ratio of capital to labour.  It is the latter ratio that is important to per capita production. Increases in per capita output depend upon the size of the capital stock relative to the labour input.  The figure plots the per capita income (the production function f(k)) on the vertical axis and the capital-labour ratio, **k**, on the horizontal axis:

The process of economic growth is the process of capital accumulation.  The rate of accumulation of capital is the rate at which the capital to labour ratio grows, dk/dt.  But before looking at the growth of the ratio, k, first consider the growth in the stock of capital, dK/dt -- not the ratio.
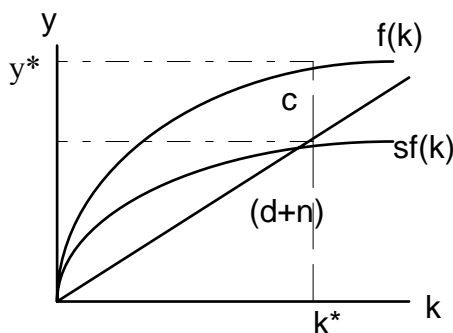


       The basis for growth is that there is positive net investment, **I$_n$**:  that the rate of investment is sufficient to replace depreciating capital, δK, and still have something left over to expand the capital stock:[1]

$$\frac{dK}{dt} = I_n = I_g - \delta K = sY - \delta K$$

where **I$_g$** indicates gross investment. Recall that in equilibrium savings will equal investment.  In this case saving is sY.[2]

       But since the population is growing, looking at the growth of the total capital stock is not enough. Now look to the rate of change of the per capita stock of capital: k.  By definition, the rate of change in the ratio, k, is dk/dt.  This can be written in a simpler way (eventually!):

$$\frac{dk}{dt} = \frac{d\left(\frac{K}{L}\right)}{dt} = \frac{L\frac{dK}{dt}}{L^2} - \frac{K\frac{dL}{dt}}{L^2} = \frac{\frac{dK}{dt}}{L} - k\left(\frac{1}{L}\frac{dL}{dt}\right),$$

Let **n**=(1/L)(dL/dt) be the percentage growth rate in the labour force per year.  We will consider **n** an exogenous variable -- we do not explain the population growth rate in this model.  But substituting from the first equation for dK/dt, and noticing that (Y/L)=f(k), we now have:

$$\frac{dk}{dt} = \left(\frac{(sY - \delta K)}{L}\right) - kn = s\frac{Y}{L} - \delta k - nk = sf(k) - (\delta + n)k$$

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[1]We assume that depreciation is simply a percentage, δ, of the capital stock, K.  So that annual depreciation takes place in amount δK.
[2]Since the consumption function is C=a+bY; and Y-C=S, clearly S=-a+(1-b)Y, or for our purposes, sY.

The final part of the expression is the form that fits with the picture. The capital -labour ratio will increase so long as the amount of saving available is greater than what is necessary to replace the capital stock, which is deprecitating at rate δ, and the new population -- which is growing at rate, n.[3]

In the figure, the rate of savings (and investment) is represented by the curve, sf(k), while the straight line has the slope (n+δ). (Because I can't put a δ in the figure, it appears in the figure as n+d.) Where they intersect is the steady state -- at k* in the figure. At the steady state, per capita income is constant, but total output is growing at a rate sufficient to replace depreciating capital and equip each new member of the labour force with the steady state per capita amount of capital -- that is why the ratio of capital per head is constant at k*.

To the left of k*, the capital to labour ratio, and per capita income, is always growing as the level of saving (and investment), sf(k), is greater than the rate of population increase and (plus) the rate of depreciation, (n+δ)k. Thus the ultimate check on the rate of growth in this framework is that the capital stock eventually rises by so much that all the savings is absorbed replacing the depreciation and equipping each new member of the population with the same stock of capital as everyone else. At k*, the level of income in this steady state is f(k*). Since savings is sf(k*), consumption is what is left over, f(k*)-sf(k*), or what is marked as **c** on the diagram.

Techncial progress. Technical progress will shift the f(k) curve up so that a new higher steady state stock of capital (and income) is possible.

**Exercises:**
(1) Draw the situation in which the rate of population growth declines from the value **n** to **n₁**.
(2) If technical change occurs so that production is now 10% more efficient than before, what is the effect on the steady state stock of capital? (I.e. does it rise or fall?) Draw the picture and explain the process by which we get to the new steady state stock of capital -- call it k**. Compared to consumption at k*, is consumption higher or lower in the post technical progress situation?
(3) How does a model of this sort change your views about economic growth and why countries grow as they do?

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[3]If there are new people being added to the population at every instant at rate **n**, then each new person has to be equipped with the same per capita stock of capital if the economy-wide ratio of capital to labour is to be maintained. Without any savings, each addition to the population tends to reduce the ratio of capital to labour.