

### Manipulating Supply and Demand

When we observe a set of prices and quantities we have no more than a set of points in a price-quantity space. In order to make sense of the data economists often use simple supply and demand analysis. Converting the theory that we know into a form that will allow us to answer the various questions we wish to ask is sometimes less obvious than it might appear at first blush. In this handout we examine a simple log-linear demand and supply system to show some of the ways in which the model may be used.

We begin by representing the demand schedule as the quantity demanded equal to a function of the own price, P, and "all other influences," D. Since the system is log-linear we have equation 1:

$$1. \quad Q^d = DP^{-\varepsilon}$$

where  $\varepsilon$  is the constant own price elasticity of demand.<sup>1</sup>

In a similar way we can express a log-linear representation of the supply schedule where the quantity supplied is written as a function of the own-price and all other "supply-side influences," S. This is equation 2 in which  $\gamma$  is the elasticity of supply.

$$2. \quad Q^s = SP^\gamma$$

To close the system we need an assumption about equilibrium. In this

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<sup>1</sup> Notice that we are not asserting constant price elasticities in general. For most work, however, it is frequently assumed to be true in the neighbourhood of the means being examined empirically. Obviously, for certain kinds of models, e.g. revenue maximizing tariff rates, or optimum revenue from money growth, this would not be an appropriate form. If the elasticity of demand is constant, however, then the reader is invited to show that the demand schedule will have the form of equation 1.

(Hint: Recall that for the constant elasticity case,  $\varepsilon = -\frac{dQ/Q}{dP/P}$  and therefore,  $-\varepsilon \frac{dP}{P} = \frac{dQ}{Q}$ . Recall

from the definition of the log-derivative, or the hat calculus, that  $\hat{x} = d \ln x = \frac{dx}{x}$ . Thus

$-\varepsilon d \ln P = d \ln Q$ . Integrate both sides and observe the result.  $-\varepsilon \int d \ln P = \int d \ln Q$  or

$-\varepsilon \ln P + D' = \ln Q$ , where  $D'$  is the constant of integration. Exponentiating both sides and letting  $\exp(D')=D$  yields the result:  $DP^{-\varepsilon} = Q$

case we assume that the quantity demanded equals the quantity supplied,

$$3. \quad Q^d = Q^s$$

which will be simply referred to as  $Q$ , the equilibrium quantity.

### The Reduced Form

With this simple framework in hand we can solve for the reduced form of the system. That is, we can solve for both the equilibrium price,  $P$ , and the equilibrium quantity,  $Q$ , the endogenous variables, as functions of  $\underline{D}$  and  $\underline{S}$ , the exogenous variables of the system. Solving first for price we have:

$$4. \quad P = \left( \frac{D}{S} \right)^{\frac{1}{\varepsilon + \gamma}}$$

and then solving for quantity,

$$5. \quad Q = D^{\frac{\gamma}{\varepsilon + \gamma}} S^{\frac{\varepsilon}{\varepsilon + \gamma}}$$

In Figure 1, equations 4 and 5, with the appropriate values for the demand and supply elasticities and the exogenous variables, represent the equilibrium point,  $\underline{E}$ .

But in most cases, merely representing the equilibrium is not enough. We also need to describe the comparative static results of the model. And it is here that the log-linear form is particularly helpful.

### Comparative Statics

To describe the comparative static changes that take place, recall that we are attempting to describe the behaviour of the endogenous variables resulting from small changes in the exogenous variables. In our case this amounts to differentiating equations 4 and 5. To make things easy, take the natural logs of equation 4,

$$6. \quad \ln P = \frac{1}{\varepsilon + \gamma} (\ln D - \ln S)$$

and then differentiate:

$$6.1 \quad d \ln P = \frac{dP}{P} \equiv \hat{P} = \left( \frac{1}{\varepsilon + \gamma} \right) (d \ln D - d \ln S) = \left( \frac{1}{\varepsilon + \gamma} \right) (\hat{D} - \hat{S})$$

where the " $\hat{\phantom{x}}$ " is the percentage change operator so that  $d \ln x = \frac{dx}{x} = \hat{x}$ .<sup>2</sup>

How well does equation 6.1 conform to our expectations? If we shift the demand schedule up and to the right, D1 D1 in Figure 2, then with  $\hat{D} > 0$ , we would expect price to increase. The exact amount price increases depends on both the elasticities of supply *and* demand. A shift in the demand schedule implies that price will increase along the supply schedule, and any movement in price will also be reflected in movement along the demand schedule. Similarly, an increase in supply will reduce the price in proportion to as well.

Changes in the equilibrium quantity can be found by taking the (natural) logarithm of equation 5,

$$7. \quad \ln Q = \left( \frac{\gamma}{\varepsilon + \gamma} \right) \ln D + \left( \frac{\varepsilon}{\varepsilon + \gamma} \right) \ln S$$

and differentiating. This leaves an expression in percentage change form, 7.1:

$$7.1 \quad \hat{Q} = \left( \frac{\gamma}{\varepsilon + \gamma} \right) \hat{D} + \left( \frac{\varepsilon}{\varepsilon + \gamma} \right) \hat{S}$$

This, too, conforms to our expectations as positive shifts in either demand or supply lead to increases in the equilibrium quantity produced and consumed. The exact amount of the quantity change depends upon an elasticity-weighted average of the forces  $\hat{D}$  and  $\hat{S}$  affecting the model.

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<sup>2</sup> To show that  $d \ln x = (dx/x)$ , recall the definition of the function,  $e^x$ . It is defined as the function for which the derivative is the same as the function. That is,  $\frac{d(e^x)}{dx} = e^x$ . Now, let  $y = e^x$ . Differentiate so that

we see:  $\frac{dy}{dx} = e^x$ , or  $\frac{dy}{e^x} = dx$ . Go to the original  $y = e^x$  and take the logs, so that  $\ln y = x$ . Now

differentiate this expression,  $d \ln y = dx$  and substitute for  $dx$ , so that we have  $d \ln y = dx = \frac{dy}{e^x} = \frac{dy}{y}$ .

Thus  $d \ln y = \frac{dy}{y} \equiv \hat{y}$ .

## Some Extreme Cases

Now equations 6.1 and 7.1 are appropriate reduced forms for most values of the demand and supply parameters,  $\alpha$ , and  $\beta$ . But if either demand or supply become perfectly elastic, then the reduced form must be different.

Consider the case in which the demand schedule is perfectly elastic.<sup>3</sup> In this case, price,  $P$ , is exogenous, and only the supply schedule, equation 2, is relevant for the determination of the equilibrium quantities. Thus equilibrium quantities are determined as equation 8

$$8. \quad \hat{Q} = \hat{S} + \gamma \hat{P}$$

A characterization such as this might be appropriate, for example, in an international setting in which producers in the home country face a large world market which set the price of the good they are producing. Notice that in estimating quantity, the appropriate regression is OLS since price is no longer an endogenous variable.

Consider another extreme. Let supply be perfectly elastic. In this case the determination of price is once again exogenous, but the appropriate characterization of quantity is through equation 1, the demand equation so that in percentage change form,

$$9. \quad \hat{Q} = \hat{D} - \epsilon \hat{P}.$$

Once again the appropriate econometric technique is OLS but this time it is the demand schedule that is identified.

If both price and quantity are determined simultaneously within the model, i.e. both are endogenous variables as is implicitly assumed in equations 1-7, then any regression on equation 2 alone would have to be two-stage to account for the endogeneity of prices.

## What Goes Into $\underline{D}$ and $\underline{S}$ ?

Thus far we have used  $\underline{D}$  and  $\underline{S}$  to represent all the influences other than own-price effects on demand and supply. The actual components of demand shift and supply shift terms depend on the questions being asked and the data available. In general the demand shift variables depend upon the usual variables entering the demand function, income, the price of substitutes and complements, and whatever special constraints might be in effect -- a tariff, tax or whatever. Similarly, on the supply side, if we think of ourselves as looking at a model for an industry, then the ingredients of  $\underline{S}$  are determined by the constraints faced by firms in the industry. In what follows we will look at some simple

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<sup>3</sup> What this means is that the elasticity of demand approaches infinity, or at least is very large.

representations of both the demand side and the supply side to capture what is subsumed under the D and S rubrics.

### Demand

The arguments of D include everything other than the own price. In particular, such things as income and the prices of substitutes and complements should be Included. One representation of such a model is displayed in equation 10

$$10. \quad D = \prod_i P^{\varepsilon_i} P^{\varepsilon} y^{\varphi}$$

In this case there is an income term, and an income elasticity, P, and a series of  $i$  goods which are presumably substitutes and complements depending on the signs of each cross-price elasticity, as well as the familiar own price elasticity which is without a subscript.

### Supply

The variables included in 5 depend upon both the form of the production technology that is assumed and the choices available to the firm or industry in factor markets. For example, if we are interested in a world in which a competitive industry is producing according to a Cobb-Douglas production function, then the analysis would proceed along the following lines.

#### *Long Run*

Suppose the industry in question uses labour and capital which are in elastic supply at current wages and rentals.<sup>4</sup> Then profit maximizing behaviour requires certain first order conditions, and these can be manipulated to give a supply schedule.

Let the production technology be characterized as Cobb-Douglas as in equation 11

$$11. \quad Q = AK^{\alpha}L^{\beta}$$

As is usual, let K be the stock of capital and L the amount of labour used in production. Since the production technology is known, constrained profit maximization establishes the Lagrangian, V, written as 12.

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<sup>4</sup> This may be thought of as pertaining to a partial equilibrium approach which is useful for many applications. A general equilibrium approach in which the amount of industry output has an effect on factor rewards will be discussed in the next lecture.

12. Maximize  $V = PQ - wL - rK - \lambda(Q - AK^\alpha L^\beta)$

with respect to  $L, K, Q, \lambda$ .

The first order conditions are defined with respect to the choice of labour, capital and output since they are all variable in the long-run. They are displayed in equations 13-15, and since the constraint is assumed to bind, we have equation 16 as well.

13. 
$$\frac{\partial V}{\partial K} = -r + \lambda\alpha AK^{\alpha-1}L^\beta = 0$$

14. 
$$\frac{\partial V}{\partial L} = -w + \lambda\beta AK^\alpha L^{\beta-1} = 0$$

15. 
$$\frac{\partial V}{\partial Q} = P - \lambda = 0$$

16. 
$$\frac{\partial V}{\partial \lambda} = Q - AK^\alpha L^\beta$$

These may be solved for the familiar relation that the nominal factor reward equals the value of the marginal product, etc., by using equation 15 in 13 and 14.

But as we are interested in solving for the supply schedule which relates the profit maximizing level of output,  $Q$  to own-price,  $P$ , and the other exogenous variables that are impounded in 5, we use the first order conditions to obtain expressions in price and output.

13' 
$$K = \alpha \frac{PQ}{r}$$

14' 
$$L = \beta \frac{PQ}{w}$$

The expressions for  $L$  and  $K$  from equations 13' and 14' may be substituted into 16 to obtain a reduced form. That is, by eliminating the stocks of capital and labour (our

endogenous choice variables), we are left with a supply schedule, a relationship between P and Q, parameters of the technology, A,  $\alpha$  and  $\beta$ , and the parameters of the marketplace -- w and r, as we wished:

$$17. \quad Q = A \left( \alpha \frac{PQ}{r} \right)^\alpha \left( \beta \frac{PQ}{w} \right)^\beta = A \alpha^\alpha \beta^\beta (PQ)^{\alpha+\beta} r^{-\alpha} w^{-\beta}$$

But if the production technology is such that  $\alpha + \beta = 1$ , then 17 reduces to a function of P and the exogenous parameters of the system:

$$17' \quad P = A^{-1} \alpha^{-\alpha} \beta^{-\beta} r^\alpha w^\beta$$

Surprisingly, perhaps, the level of output, Q, does not appear. What this means is that the supply schedule is perfectly elastic at a level determined by the technology and costs associated with production. In such a situation, demand can only determine quantity. In this case the supply schedule is horizontal as  $\epsilon$  is infinite.

### *Short Run*

Suppose now we consider the case in which there is a short run in the sense that not all factors are variable. For example, if capital is assumed (arbitrarily) to be fixed, then our profit maximizing solution involves only choices over Q, L, and A. In this case, of course, the return to capital is a rent determined by the value of the marginal product schedule.

Formally, the first order conditions look like equations 14-16, and as before, we can make use of 14' to solve for the endogenously chosen stock of labour utilized. Substituting into equation 16, however, we are left with an expression in the various parameters of the system and the capital stock:

$$18. \quad Q = AK^\alpha \left( \beta \frac{PQ}{w} \right)^\beta$$

or,

$$18' \quad Q = A^{\frac{1}{\alpha}} \beta^{\left(\frac{\beta}{\alpha}\right)} w^{-\left(\frac{\beta}{\alpha}\right)} K P^{\left(\frac{\beta}{\alpha}\right)}$$

Thus we have an expression in which the components of S, the supply shift parameter is identified in terms of the technology and exogenous factor reward or factor quantity. Interestingly, the elasticity of the supply schedule,  $\epsilon$ , is a function of the underlying technology which is also the ratio of factor shares. In this case the supply schedule is upward sloping as a function of price. Since factors are not all freely available at

constant factor prices, any expansion of output runs into diminishing returns. Increases in the capital stock will raise output. Increases in the wage will shift the supply schedule up, and technical change will be reflected as a shift in supply as well.