Multiple Use Values and Convergence of Optimal Harvesting Policies

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Abstract

The concept of sustained yield management for renewable resources has a long history. It is justified on the grounds of maintaining now and in the future their ecological, economic and social functions. Now, Heaps (2015) has rigorous economic justification for this concept in the context of the management of uneven aged forests. He identifies the optimal steady state age distributions (as normal forests) and was able to show that optimal harvesting policies generate age class distributions that converge over time to such steady states. However, he considers only the wood values and not the non-timber values provided by forests. This paper shows that under certain assumptions the convergence result can also hold when both types of values are jointly included in the model.

Keywords: multiple use forestry, multiple age classes, optimal harvesting, convergence
1. Introduction

The concept of sustainable forest management goes back to at least von Carlowitz (1713), the father of sustained yield. Today it means determining how to use forests to fulfill, now and in the future, their ecological, economic and socio-cultural functions.¹ Forest managers should plan how to use forests today to ensure that similar benefits, health and productivity accrue to the future. In the 18ᵗʰ century, this led to the concept of the normal forest as a desirable target for the the age class distribution of forests. A formal economic justification of this target has, however, has not been fully provided until Heaps (2015). Heaps (1984, 2015) developed a model for describing the optimal harvesting of forest land which is covered by trees varying in ages. There is a large literature on such models which is reviewed in the second paper. The focus has been on the value of the logs harvested from such forests. However, forests also provide amenities, non-timber products and other economic values before they are harvested². The significance of these non-timber values for the harvesting problem was recognized by Hartman (1976) and many other later papers³. Hartman’s model is a single stand model where all the trees are of the same age. Recently, his model has been extended to multiple age class forests in papers by Heaps (1995), Sahashi (2002), Tahvonen (2004a, 2004b, 2009) and Uusivuori and Kuuluvainen (2005, 2008).

Heaps (1995) develops a multiple use forestry maximum principle (MUFMP) that the optimal policy must satisfy when both timber and non timber values are considered. The purpose of this paper is to show that his proof of convergence can be extended to this version of the multiple use value model provided appropriate assumptions are made.⁴ In contrast, Sahashi (2002) also has a convergence result when the amount of land used for growing trees is a choice variable. In Tahvonen (2004b), non-timber values depend on the fraction of the land preserved as old growth. The long run age distribution depends on the initial age class distribution as a continuum but a numerical example suggests convergence to a normal forest. In Uusivuori and Kuuluvainen (2008), there is a second asset as well as the forest land. The long run age distribution depends on the the distribution of wealth between the two assets.

The paper is organized in a manner similar to Heaps (2015). Section 2 offers

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¹See the sustainable forest management article in Wikipedia (https://en.wikipedia.org/wiki/Sustainable_forest_management)


⁴Heaps (1995, 330) measures timber values as the $ value of the logs harvested while Heaps (2015, 76) uses instead the utility (in $) of the volume harvested. In the multiple use case, some aspects of the proof of convergence do not easily generalize in the second model. So here the simpler model is used which also simplifies some parts of the convergence proof.
a new proof that the optimal logging policy logs the oldest trees standing in the forest first. A condition is then given which ensures that an optimal harvesting policy actually involves cutting down some trees but does not involve cutting down very young trees. The second condition is required in the proof of convergence. Harvesting policies are then represented by functions $h \in L^1_1(e^{-rt})^5$ and the multiple use multiple age class model is described. Section 3 deals with the existence of optimal harvesting policies. Section 4 reviews the MUFMP derived in Heaps (1995) and derives the integral form of the adjoint equation for the multiple use model. Section 5 is concerned with sequences of optimal harvesting policies that converge in the weak topology$^6$. It provides a condition under which the associated adjoint variables converge pointwise a.e.. Section 6 then shows that under this and an additional condition the policies themselves converge pointwise a.e. and in PV to an optimal policy. It then discusses the proof of asymptotic stability which is mainly just the same as the similar proof in Heaps (2015). However, the steady state limit is now given by a modified Hartman formula. Finally, Section 7 concludes the paper.

2. The multiple use forestry age class model in continuous time

It is desired to devise an optimal logging policy for a forest of total area $A$ which is homogeneous with respect to its biological and economic characteristics. Initially, the area is covered by an uneven aged forest which has $h_0(a)$ hectares of $a$ year old trees in each of the age classes $0 \leq a \leq a_o$ where $\int_0^{a_o} h_0(a)da = A$. Then as time passes the age distribution of the forest changes either by trees becoming older or by land being logged and immediately reforested so that it is now is covered by $0$ age trees. The process of age distribution change can be captured by the following model well known in population dynamics. Let $H(t,a)$ be the number of hectares (ha.) covered by $a$ year old trees at time $t$ and $h(t,a)$ the number of these ha. logged at time $t$. The age distribution change is governed by the partial differential equation

$$\frac{\partial dH}{\partial t}(t,a) + \frac{\partial dH}{\partial a}(t,a) = -h(t,a)$$

(see Quinn (1992), for example). This equation has a unique solution given a logging policy and the initial age distribution $H(0,a) = h_0(a)$.

The values generated by such a policy are as follows. The revenue per ha. associated with logging land covered by a year old trees will be described by a function $p(a)$. It will be assumed that this revenue function satisfies the following properties.

(a) $p(a) \geq 0$ for all $a$.

(b) $\int_0^a p(a)da > 0$ when $p(a) > 0$ and $\lim_{a \to \infty} p(a) = p < \infty$.

These properties are similar to those discussed in Heaps (1984). At time $t$ the logging policy will generate timber revenues of $\int_{a=0}^\infty p(a)h(t,a)da$. In addition,
as in Hartman (1976), it will be assumed that land not being logged at time \( t \) provides other services valuable to society which can be measured in $ terms by a Lesbegue measurable non-timber value function \( E(a) \) per ha. of \( a \) year old trees. These non-timber values will be summed across the different ages of standing trees at time \( t \) so are in total
\[
\int_{a=0}^{\infty} E(a)H(t,a)da.
\]
Logging costs will be assumed to take the form
\[
C\left(\int_{a=0}^{\infty} h(t,a)da\right)
\]
where \( \int_{a=0}^{\infty} h(t,a)da \) is the area of land logged at time \( t \). The cost function will be assumed to be \( C^2 \) and to satisfy

(c) \( C(0) \geq 0 \) and \( C \) is strictly increasing in \( h \).

(d) \( C(h) \) is strictly convex in \( h \). Thus \( AVC(h) = (C(h) - C(0))/h \) has a minimum value of \( C'(0) \) occurring at \( h = 0 \) because \( AVC'(h) > 0 \) for all \( h > 0 \).

Optimal logging policies will maximize the present value of timber revenues plus non-timber values generated by the policy net of the logging costs associated with the policy. At this point, it is worthwhile to identify conditions under which such a logging program will actually involve cutting down some trees.\(^7\) As in Hartman (1976), consider a plot of bare land which is planted at time \( 0 \) and then can grow as an even aged forest. Suppose there is no future harvest of this land. The present value (per ha.) of the non-timber services generated by this land per ha. will be
\[
\overline{W}_{NT} = \int_{u=0}^{\infty} E(u)e^{-ru}du.
\]
Clearly if \( \overline{W}_{NT} = \infty \), this is the optimal policy. Thus it will also be assumed that
\[\text{(e) } \overline{W}_{NT} < \infty \text{ so that } E \in L^1_e(e^{-rt}).\]
Now suppose the plot is harvested once at age \( a \) and rate \( h \) and then regenerated and allowed to grow ever thereafter without further harvest. The present value per ha. generated by this policy is
\[
V(a) = \left( p(a) - (C(h) - C(0))/h \right)e^{-ra} + \int_{u=a}^{\infty} E(u)e^{-ru}du + e^{-r\overline{W}_{NT}}
\]
\[
\dot{V}(a) = \left( p(a) - r(p(a) - (C(h) + C(0))/h) + E(a) - r\overline{W}_{NT} \right)e^{-ra} \quad (1)
\]
Note that \( V(0) = -(C(h) - C(0))/h + \overline{W}_{NT} \leq \overline{W}_{NT} \). Thus \( \underline{a}_h = \text{glb}\{a : V(a) > \overline{W}_{NT}\} > 0 \) if \( h > 0 \) and harvesting the plot can only be optimal if \( \underline{a}_h < \infty \). If \( h = 0 \), \( \underline{a} = \underline{a}_0 > 0 \) as well provided \( C'(0) > 0 \) or provided \( E(a) - r\overline{W}_{NT} < 0 \) for small \( a \) so that \( V(a) < 0 \) for these \( a \).\(^9\) Logging the plot when the trees are of age \( a \) where \( a < \underline{a} \) cannot be economically optimal as a larger present value can be obtained by never logging this plot or at least by deferring the logging to a later date. Now note that \( V(a) \) is a decreasing function of \( h \) by assumption (d) so harvesting the plot will be optimal for some \( h \) only if \( \underline{a} < \infty \). Moreover, since \( \lim_{a \to \infty} V(a) = \overline{W}_{NT} \), this implies for such \( h \) that \( V(a) \) is maximized by a finite \( a \) given \( h \). Further, from (1),
\[
\dot{V}(a) = -rV(a) + (p'(a) - rp(a) + E(a))e^{-ra}.
\]
The following assumption will thus be made.

\(^7\)This issue was not considered by Hartman (1976).

\(^9\)Any cost that would be incurred in any period in which there was no logging \((C(0))\) has been netted out of this and the following calculations.

\(^9\)If \( p(a) - C'(0) + \overline{W}_{NT} > e^{ra}\int_{u=0}^{\infty} E(u)e^{-ru}du \) for some \( a \) then \( \underline{a} < \infty \). In case \( \lim_{a \to \infty} E(a) \) exists, then by L'Hôpital's rule, \( \lim_{a \to \infty} e^{ra}\int_{u=0}^{\infty} E(u)e^{-ru}du = (1/r) \lim_{a \to \infty} E(a) \). Thus a condition which ensures that an optimal logging policy involves harvesting some trees is \( p - C'(0) + \overline{W}_{NT} > (1/r) \lim_{a \to \infty} E(a) \).
(f) \(0 < a < \infty\) and \(\dot{p}(a) - rp(a) + \dot{E}(a) < 0\) for \(a > a_h\).

This implies that \(V(a)\) has at most one local extremum for \(a > a_h\), which if it exists gives the global maximum of \(V(a)\).

In addition (f) implies the following result which makes it possible to present optimal logging policies in a tractable manner.\(^{10}\)

**Proposition 1.** At any time, an optimal logging program involves at that time cutting trees which are at least as old as any trees left standing in the forest at that time.

**Proof:** Suppose it is optimal to harvest \(\Delta h\) hectares of trees of age \(a_1\) at time \(t\), to replant this land immediately and then let the trees on it grow until time \(t + b_1\) when this land is harvested again. Further, suppose another \(\Delta h\) hectares of trees of age \(a_2 + dt\) are harvested at time \(t + dt\) where \(dt > 0\). The land is replanted immediately and then the next harvest takes place at time \(t + dt + b_2\). This logging program can be compared to a program which is identical except in that the order in which the two plots above are harvested is reversed and the times of the next harvests of these plots are also reversed. The switch does not change harvesting costs. At time \(t\), the value of the harvest changes by \((p(a_2) - p(a_1))\Delta h\) and at time \(t + dt\) it changes by \((p(a_1 + dt) - p(a_2 + dt))\Delta h\). The present value of the environmental services provided by the first plot between planting and harvesting at time \(t\) was \(e^{r(a_1 - t)} \int_{0}^{a_1} E(u)e^{-ru}du\) which changes under the switch to \(e^{r(a_1 - t)} \int_{0}^{a_1 + dt} E(u)e^{-ru}du\) and for the second plot the net change is \(-e^{r(a_2 - t)} \int_{a_2}^{a_2 + dt} E(u)e^{-ru}du\).\(^{11}\) Overall then the present value of the net benefits per ha. from the harvesting program is essentially changed only by

\[
\Delta e^{-rt} = \left(p(a_1 + dt) - p(a_2 + dt)\right)e^{-r(t + dt)} + \left(p(a_2) - p(a_1)\right)e^{-rt}
\]  

(2)

Now suppose \(a_1 < a_2\) Then

\[
d\Delta /dt = \left(\dot{p}(a_1 + dt) - rp(a_1 + dt) - \dot{p}(a_2 + dt) + rp(a_2 + dt)\right)e^{-r(dt)}
\]

By assumption (f) about the value functions, it follows that \(\dot{p}(a) - rp(a) + E(a)\) is a decreasing function of \(a\) for \(a > a_h\). Thus the above derivative is

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\(^{10}\) This result is explained indirectly in Heaps (1995).

\(^{11}\) The first plot under the optimal harvesting program also provided a present value of services from time \(t\) to time \(t + b_1\) of \(e^{-rt} \int_{0}^{a_1} E(a)e^{-ra}da\). This is lost with the reversal but there is a gain of \(e^{-r(t+dt)} \int_{0}^{b_2} E(a)e^{-ra}da\) from time \(t + dt\) to time \(t + dt + b_2\). These changes for the second plot are the same but in the reverse order.
positive and \( (p(a_1 + dt) - p(a_2 + dt))e^{-rt}dt + e^{ra_2} \int_{a_1}^{a_1+dt} E(u)e^{-ru}du > p(a_1) - p(a_2) + e^{ra_2} \int_{a_1}^{a_1+dt} E(u)e^{-ru}du \) and since \( \Delta = 0 \) when \( dt = 0 \), the switch of plots increased the present value of the logging policy and so the original order was not optimal. It must have been the case then that \( a_1 \geq a_2 \) which proves the proposition.

Based on Proposition 1, the search for optimal harvesting policies can be restricted to those policies which cut down the forest in the same order as it has been planted. The combination of an initial age distribution and such a logging policy can then be represented by a single function \( h(t) \) in \( L^1_{\infty}(e^{-rt}) \). For \( 0 \leq a < a_o \), the initial age distribution is \( h_o(a) = h(a_o - a) \) and \( A = \int_0^{a_o} h(a_o - a)da \).

This part of \( h \) may take on the value \( +\infty \). For \( t \geq a_o \), \( h(t) \) is the number of hectares that will be logged (and regenerated) at time \( t - a_o \) after the base time \( a_o \). Given \( h \), \( v(t) \) and \( w(t) \) are defined as the time when the trees being harvested at time \( t \) were planted and as the time at which the land being cleared at this time will next be harvested again (see Heaps (2015, 77)). Using the base time \( a_o \), the present value of revenues from selling the harvested logs net of harvesting costs is

\[
W_T(h) = e^{ra_o} \int_{t=a_o}^{t=\infty} [p(t - v(t))h(t) - C(h(t))]e^{-rt}dt \tag{3}
\]

There is also the present value of the environmental services provided by the standing forest after time 0. This can be written in either of the following two ways as\(^{12}\)

\[
W_{NT}(h) = e^{ra_o} \int_{t=0}^{t=\infty} \int_{u=0}^{u=w(t)-t} E(u)e^{-ru}du[h(t)e^{-rt}dt \tag{4a}
\]

\[
W_{NT}(h) = e^{ra_o} \int_{t=a_o}^{t=\infty} \int_{u=0}^{u=t-v(t)} E(u)e^{-ru}du[h(t)e^{-rv(t)}dt \tag{4b}
\]

One further assumption will be made to avoid having to consider impulsive controls as these would require a more complicated type of model involving generalized differential equations (Miller and Rubinovich (2003)). The condition

\[
(g) \quad \bar{p} + W_{NT} \leq \lim_{h \to \infty} C'(h)
\]

achieves this purpose. Then for \( p < \bar{p} \), \( (p + W_{NT})h - C(h) + C(0) \) is maximized by a finite \( h > 0 \). Consequently, the search for optimal harvesting policies can be restricted to policies which satisfy \( h(t) < \infty \) for all \( t \geq a_o \).

Thus the problem of maximizing the present value of net benefits from logging in the multiple use forestry age class model in continuous time can now be presented as choosing \( h \in L^1_{\infty}(e^{-rt}) \) to maximize

\[
W(h) = W_T(h) + W_{NT}(h)
\]

\(^{12}\)This calculation will include some environmental services which occur before the base time \( t = a_o \) but this does affect the comparison of policies based on the same initial age distributions. See Heaps (1995, 333,337) for a discussion of these formulae.
\[ e^{r_0} \int_{t=a_o}^{\infty} [p(t-v(t))h(t))-C(h(t))]e^{-rt} dt + e^{r_0} \int_{t=0}^{\infty} h(t)e^{-rt} \int_{u=0}^{w(t)-t} E(u)e^{-ru} du \] dt

(5)

The search for an optimal policy can be restricted to \( \mathcal{A} \subset L^1_c(e^{-rt}) \) whose members satisfy the following conditions\(^{13}\).

\( \begin{align*}
(a) & \quad v(t) \text{ and } w(t) \text{ are defined as in Heaps (2015)} \\
(b) & \quad h_o(a) = h(a_o - a) \text{ for } 0 \leq t \leq a_o \text{ is given} \\
(c) & \quad h(t) < \infty \text{ for } t \geq w(0) \\
(d) & \quad a \leq t - v(t) \text{ for } t \geq w(0)
\end{align*} \)

The next section shows that the multiple use model (5) always has a solution.

3. Existence of an optimal harvesting policy

Existence of an optimal harvesting policy considering only the timber value part \( W_T \) of \( W \) is shown in Heaps (2015). The arguments and results presented there apply also to \( W \) if \( W_{NT} \) is bounded above on \( \mathcal{A} \) and if \( \lim_{n \to \infty} h_n = h^* \) in the weak topology implies \( \lim_{n \to \infty} W_{NT}(h_n) = W_{NT}(h^*) \).

The argument for the boundedness of \( W_T \) presented there simplifies to

\[ W_T(h) \leq e^{r_0} \int_{a_o}^{\infty} [(p - C'(0))h - C(0)]e^{-rt} dt \]

and uses \( C(h) \geq C'(0)h + C(0) \) (by strict concavity of \( C \)) and \( \int_{a_o}^{\infty} he^{-rt} dt \leq e^{-r_0} A/(1 - e^{-r_2}) \)\(^{14}\). Further

\[ W_T(h_n) - W_T(h^*) \leq e^{r_0} \int_{a_o}^{\infty} \{(c_n - c^*) - C'(h^*)(h_n - h^*)\}e^{-rt} dt \]

where \( c(t) = p(t - v(t))h(t) \)\(^{15}\).

The limit as \( n \to \infty \) of the RHS is 0 given the weak convergence of the \( \{c_n\} \) and \( \{h_n\} \) to \( c^* \) and \( h^* \) provided \( C'(h^*) \) is bounded above. Thus \( \limsup_{n \to \infty} W_T(h_n) \leq W(h^*) \).

For the boundedness of \( W_{NT} \), from (4a), \( W_{NT}(h) \leq e^{r_0} W_{NT}\int_{0}^{a_0} h(t)dt + \int_{a_0}^{\infty} h(t)e^{-rt} dt \) \leq W_{NT}[e^{r_0} A + A/(1 - e^{-r_2})] \) and by (e) \( W_{NT} < \infty \). For the limit

\[ |W_{NT}(h_n) - W_{NT}(h^*)| = \int_{0}^{\infty} [(h_n - h^*)e^{-rt} \int_{a=0}^{w_n(t)-t} E(u)e^{-ru} du ] + h^*e^{-rt} \int_{u=w^*(t)-t}^{w_n(t)-t} E(u)e^{-ru} du | dt \]

\(^{13}\) \( w(0) \geq a_o \) is the time at which active logging begins.

\(^{14}\) See (13) in Heaps (2015).

\(^{15}\) As shown in Heaps (2015, 84), \( \{w_n\} \) and \( \{v_n\} \) converge pointwise a.e. to \( w^* \) and \( v^* \) and \( \{c_n\} \) converges weakly to \( h^* \).
\[
\leq \int_0^\infty |h_n - h^*| W_N T e^{-rt} dt + \int_0^\infty h^* e^{-rt} \int_{u=a_{n(t)}-t}^{u=t-v(t)} E(u) e^{-ru} du dt
\]

The limit of the first term here is 0 because of (e) and the weak convergence of the \( \{h_n\} \) to \( h^* \). For the second term the limit of the integrand is 0 a.e. and is also integrable so by the Lesbegue dominated convergence theorem the limit of the integral is also 0. Thus, for the multiple use multiple age class forest, it can be concluded in the same way as in Heaps (2015, 79) that (i) and (ii) of the following lemma hold. Part (iii) uses (4b) so

\[
\text{Part (iii) uses (4b) so}
\]

\[
W(h) = e^{r a_0} \int_{a_0}^{\infty} [e - C(h) + h e^{r(t-v)} \int_{u=0}^{u=t-v(t)} E(u) e^{-ru} du] e^{-rt} dt
\]

Then \( \lim_{n \to \infty} c_n(t) - C(h_n(t)) + h_n(t)e^{r(t-v_n)} \int_{u=0}^{u=t-v_n(t)} E(u) e^{-ru} du + f = e^* - C(h^*(t)) + h^*(t)e^{r(t-v^*)} \int_{u=0}^{u=t-v^*} E(u) e^{-ru} du + f \) a.e. Fatou’s lemma is applied to this equation.

**Lemma 1.** (i) If there exists \( \tilde{h} < (h > 0) \) such that \( h_n(t) \leq \tilde{h}(t) \) \( \forall n \) and \( t \) a.e. on an interval, then \( h^*(t) \leq \tilde{h}(t) \) a.e. on this interval. A similar result applies for the \( c \)’s.

(ii) If either \( \lim_{n \to \infty} C'(h) < \infty \) or \( \{h_n\} \) is uniformly bounded above on \( [a_0, \infty) \) as in (i), then

\[
\lim_{n \to \infty} \sup W(h_n) \leq W(h^*)
\]

(iii) Under the same conditions as (ii), if \( \lim_{n \to \infty} h_n(t) = h^*(t) \) a.e. and there is an \( f \) in \( L^1_{+}(e^{-rt}) \) such that \( c_n(t) - C(h_n(t)) + h_n(t)e^{r(t-v_n)} \int_{u=0}^{u=t-v_n(t)} E(u) e^{-ru} du \geq -f(t) \) for all \( t \) and \( n \), then \( \lim_{n \to \infty} W(h_n) = W(h^*) \).

The proof of existence in Heaps (2015, 79) using Lemma 1 then continues to be valid.

**Proposition 2.** For any initial age distribution of the forest, there is a \( h \in A \) which maximizes \( W(h) \) as given by (5). Further, there is at least one initial age distribution and logging policy which yields a higher present value of the social surplus than any other initial age distribution and logging policy.

**4. The multiple use forestry maximum principle**

This principle was derived in Heaps (1995).\(^{16}\) It can be stated as follows. The current value Hamiltonian for the multiple use forestry maximum principle is

\[
J(h) = p(t - v(t))h - C(h) + h \int_0^{w(t)-t} E(u)e^{-ru} du - qh
\]

\(^{16}\)A direct proof of the two propositions in this section can be obtained from http://www.sfu.ca/~heaps/exfmp/mufmp.pdf.
where the multiplier \( q(t) \) is chosen for \( t \geq a_o \) to be an absolutely continuous function so that a.e.

\[
\dot{q} - rq = \dot{p}(w - t)e^{-r(w-t)} - \dot{p}(t - v)\dot{o} + E(w - t)\dot{w}e^{-r(w-t)} - E(t - v)
\]  

(8)

**Proposition 3.** MUFMP An optimal solution \( h \) for the multiple use multiple age class continuous model (5) maximizes the Hamiltonian \( J(h) \) in (7) subject to \( t \geq a_o \). The other conditions in (5) must also be satisfied and the multiplier \( q(t) \) (defined on \([a_o, \infty)\)) should be absolutely continuous and satisfy a.e. the adjoint equation (8).

The transversality condition \( \lim_{t \to \infty} qH(t)e^{-rt} = 0 \) should also hold.

Also useful are the integral forms of the adjoint equation which are

\[
q(t) = \int_{t}^{w(t)} [\dot{p}(s - v(s))\dot{v} + E(s - v)]e^{-r(s-t)}ds
\]

\[
= \int_{v(t)}^{w(t)} [\dot{p}(w(s) - s) + E(w - s)\dot{w}]e^{-r(w(s)-t)}ds
\]  

(9)

These formulas show that \( q(t) \) is well defined for any optimal logging policy and \( t \geq a_o \). The absolute continuity of \( q(t) \) is discussed by Jones (1993,550).

It should also be noted that the MUFMP is also sufficient as stated next in Proposition 4 provided condition (f) is satisfied. The proof is a straight forward extension of the proof of the similar proposition in Heaps (1984).

**Proposition 4.** The MUFMP is sufficient to describe the optimal solution to the multiple use multiple age class continuous model provided (f) \( \ddot{p}(a) - rp(a) + E(a) < 0 \) for all \( a > a \). Moreover, an optimal logging policy is unique.

5. Some implications of the MUFMP

The proof of convergence of the optimal logging polices depends on the following corollaries. They are proven for such policies using the MUFMP.

**Corollary 1.** (i) If \( \overline{p} + \overline{W}_{NT} < \lim_{h \to \infty} C'(h) \), there exists \( \bar{h} < \infty \) such that \( h(t) \leq \bar{h} \forall t \geq a_o \).

(ii) If \( \overline{p} + \overline{W}_{NT} = \lim_{h \to \infty} C'(h) \), there exists a function \( \tilde{h}(t) \) such that \( h(s) \leq \tilde{h}(t) < \infty \forall s \in [a_o, t] \) and \( \forall t \geq a_o \).

**Proof:** (i) \( J(h) = p(t - v(t))h - C(h) + h \int_{0}^{w(t)-t} E(u)e^{-ru}du - q(t)h < \phi(h) = \overline{p}h - C(h) + h\overline{W}_{NT} \forall h \geq 0 \). Since \( \phi \) is strictly concave in \( h \) with a maximum value, there is a \( \bar{h} < \infty \) for which \( \phi(\bar{h}) = -C(0) \) and for which \( J(h) < \phi(h) \leq \phi(0) = J(0) \) for \( h \geq \bar{h} \). This \( \bar{h} \) is therefore an upper bound on the \( h \) which maximizes \( J(h) \).
(ii) (9) shows that \( q(s) \neq 0 \forall s \in [a_n, t] \). Since \( q(t) \) is continuous in \( t \), there exists \( q(t) > 0 \) such that \( q(s) \geq q(t) \forall s \in [a_n, t] \). It follows that \( J(h(s)) = p(s - v(s))h(s) - C(h(s)) + h(s) \int_0^{w(s)-s} E(u)e^{-ru}du - q(s)h(s) < \phi h(s) - C(h(s)) + h(s)W_{NT} - q(t)h(s) = \phi(h(s)) \) where \( \phi \) is strictly concave in \( h \) with a maximum value. The rest of the proof is similar to the proof of (i).

Given \( h \in \mathcal{A} \), there may be an interval of choices for the base time \( a_n \) at which \( W(h) \) is calculated. The latest possible choice is \( w(0) \), the time at which active logging begins. This base time also gives the highest possible value of \( W(h) \) so will now always be used in the search for optimal logging policies. Suppose now that \( \{h_n\} \) is a sequence of optimal logging policies which converges weakly to a logging policy \( h^* \) (including in the initial age distributions). The next result will be used to show that \( h^* \) itself is actually also optimal.

**Corollary 2.** If there exist a function \( \tilde{h}(t) \) such that \( h_n(s) \leq \tilde{h}(t) < \infty \forall n \) and \( s \in (w^*(0), t] \), then \( \lim_{n \to \infty} q_n(t) = q^*(t) \) pointwise a.e. on \([w^*(0), \infty)\) where \( q^* \) is defined by (9) from \( h^* \).

**Proof:** Given (9), the proof of this corollary in Heaps (2015) works for the first term in the integral forms of the \( q_n \). For the second term

\[
\lim_{n \to \infty} \int_t^{w_n} E(s - v_n)e^{-r(s-t)}ds - \int_t^{w^*} E(s - v^*)e^{-r(s-t)}ds
= \lim_{n \to \infty} \int_t^{w_n} [E(s - v_n) - E(s - v^*)]e^{-r(s-t)}ds + \lim_{n \to \infty} \int_t^{w^*} E(s - v^*)e^{-r(s-t)}ds
\]

Given \( w > w^* \), for \( n \) sufficiently large \( w_n < w \). If the first of these limits is calculated as \( \int_t^w \) then it is 0 because \( E(a)e^{-ra} \) is bounded on any finite interval and the integrand \( \to 0 \) a.e.. The rest of the proof follows from \( \lim_{n \to \infty} w_n(t) = w^*(t) \) a.e..

6. Asymptotic Stability

It will be shown here that an optimal logging policy always converges over time to a constant amount of logging. First, it will be shown that the weak convergence of optimal harvesting policies can imply a stronger convergence.

**Proposition 5** If \( \{h_n\} \) is a sequence of optimal logging policies in \( \mathcal{A} \), \( h^* = \lim_{n \to \infty} h_n \) in the weak topology on \( L^1(e^{-rt}) \) (including in the initial age distributions) and there exist a function \( \tilde{h}(t) \) such that \( h_n(s) \leq \tilde{h}(t) < \infty \forall n \) and \( s \in (w^*(0), t] \), then \( h^* \) is an optimal logging policy and \( h^* = \lim_{n \to \infty} h_n \) a.e. on \((w^*(0), \infty)\). Moreover, if either \( \lim_{h \to \infty} C'(h) < \infty \) or \( \{h_n\} \) is uniformly bounded above on \([w^*(0), \infty)\) then \( W(h^*) = \lim_{n \to \infty} W(h_n) \).
Proof: The Hamiltonian for the initial conditions associated with \( h_n \) is

\[
J_n(h) = p(t - v_n(t))h - C(h) + h \int_0^{w_n(t)-t} E(u)e^{-ru}du - q_n(t)h
\]

If \( \lim_{n \to \infty} h_n = h^* \) in the weak topology, then since \( \lim_{n \to \infty} v_n = v^* \), \( \lim_{n \to \infty} w_n = w^* \) and \( \lim_{n \to \infty} q_n = q^* \) a.e. on \( (w^*(0), \infty) \) (see Corollary 2), \( \lim_{n \to \infty} J_n(h) = J^*(h) = p(t - v^*(t))h - C(h) + h \int_0^{w^*(t)-t} E(u)e^{-ru}du - q^*(t)h \) \( \forall h \) and almost all \( t \in [w^*(0), \infty) \). From the MUFMP, \( J_n(h) \leq J_n(h_n(t)) \) so \( J^*(h) \leq J^*(h^*(t)) \) where \( h^*(t) \) is any accumulation point of \( \{h_n(t)\} \). Thus \( h^*(t) \) maximizes \( J^*(h) \) and by Proposition 4 is unique. It follows that \( \lim_{n \to \infty} h_n(t) = h^*(t) \) a.e.. The proof of the rest of this proposition is based on (4b) using \( W(h) = e^{r w(0)} \int_w^\infty SS(t)e^{-r t}dt \) where the social surplus \( SS(t) = p(t - v(t))h(t) - C(h(t)) + h(t)e^{r(t-v(t))} \int_v^{t-v(t)} E(u)e^{-ru}du \geq -C(0) \) a.e. when \( h \) is an optimal policy. \( W(h^*) = \lim_{n \to \infty} W(h_n) \) follows from Lemma 1(iii).

The link between Proposition 5 and the limiting properties of a single optimal logging policy is provided by the concept of a translation of a member \( h \) of \( L^1 \). If \( \tau \in \mathbb{R} \), the \( \tau \) translation of \( h \) is \( h_\tau \) defined by \( h_\tau(t) = h(t + \tau) \). The properties of translations are as given in Heaps (2015, 81-2). An important property is that if \( h \) is optimal then so is \( h_\tau \) for the initial age distribution \( h_\tau_0(a) = h_\tau(w_\tau(0) - a) = h(w(\tau) - a) \) for \( 0 \leq a \leq w_\tau(0) = w(\tau) - \tau \). It can also be checked that \( SS_\tau(s) = SS(s + \tau) \) so that the following accounting identity continues to hold.

\[
W(h) = e^{r w(0)} \int_{w(0)}^{w(\tau)} SS(s)e^{-r s}ds + e^{r w(0) - w(\tau)}W(h_\tau) \tag{10}
\]

One useful implication of (10) is that if \( h \) is periodic of period \( \tau \) so that \( h = h_\tau \) and \( w = w_\tau \), then

\[
W(h) = \int_{w(0)}^{w(\tau)} SS(s)e^{-r s}ds \frac{e^{-r w(0)} - e^{-r w(\tau)}}{e^{-r w(0)} - e^{-r w(\tau)}}
\]

Now if \( \tau > w(0) \), it is possible to construct a periodic harvesting policy \( h_{\tau \tau} \) of period \( \tau \) using the portion of \( h \) which has \( 0 \leq t \leq \tau \). \( h_{\tau \tau} \) has the same initial age distribution as \( h \) but by Proposition 4 is not optimal unless \( h_{\tau \tau} = h_\tau = h \). Thus, if \( h_\tau \neq h \), \( W(h) > W(h_{\tau \tau}) \) which can be written as

\[
\int_{w(0)}^{w(\tau)} SS(s)e^{-r s}ds < (e^{-r w(0)} - e^{-r w(\tau)})W(h)
\]

Substituting this in (10) then shows that \( W(h) < W(h_\tau) \). Thus

**Proposition 6** If \( h \) is an optimal logging policy and \( \tau \geq w(0) \), then \( W(h) \leq W(h_\tau) \) where the inequality is exact if \( h_\tau \neq h \). Moreover, \( \bar{W} = \sup \{W(h_\tau) : \tau \in \mathbb{R}_+\} = \lim_{\tau \to \infty} W(h_\tau) \).
Proof: The second part follows from \((h_{\tau})_{\sigma} = h_{\tau+\sigma}\) so that \(W(h_{\tau}) \leq W(h_{\tau+\sigma})\) when \(\sigma \geq w(\tau) - \tau\). Thus \(W(h_{\tau}) \leq \liminf_{\sigma \to \infty} W(h_{\sigma})\) \(\forall \tau \geq w(0)\) which can only be the case if \(\lim_{\tau \to \infty} W(h_{\tau})\) exists and equals \(\overline{W}\).

Corollary 3 If \(h\) is an optimal logging policy and \(W(h) = W(h_{\tau_i})\) for \(i = 1, 2\) where \(\tau_i \geq w(0)\) and \(\tau_i/\tau_2\) is irrational, then \(h\) is a constant function. Thus, if \(h^*\) maximizes \(W(h)\) for all initial age distributions, then \(h^*\) is a constant function. Moreover, if \(W(h_{\tau}) \leq W(h_{\tau^*})\) \(\forall \tau \geq \tau^* < \infty\), then \(h_{\tau^*}\) is a constant function.

Proof: From Proposition 6, \(h_{\tau_1} = h = h_{\tau_2}\) which is not possible unless \(h\) is constant.

Turning now to the situation of an optimal logging policy \(h\) for a predetermined initial age distribution, let \(\{\tau_n\}\) be any increasing unbounded sequence. By the results of Section 3, this sequence can be refined so that \(\lim_{n \to \infty} h_{\tau_n} = h^*\) in the weak topology. By Corollary 1, either \(\lim_{n \to \infty} C'(h) < \infty\) or \(h\) is bounded above so \(\{h_{\tau_n}\}\) is uniformly bounded above. Thus

**Proposition 7** If \(h\) is an optimal logging policy, any increasing unbounded sequence \(\{\tau_n\}\) has a subsequence such that \(\lim_{n \to \infty} h_{\tau_n} = h^*\) a.e. on \((w^*(0), \infty)\) where \(h^*\) is an optimal logging policy with \(W(h^*) = \overline{W}\). Moreover \(h^*\) is a constant function.

Proof: By the results of Section 3, there is a subsequence which has a limit \(h^*\) in the weak topology. By corollary 2, the conditions of Proposition 15 are met. Thus the \(h^*\) which is the limit a.e. of the subsequence is optimal. Now if \(\tau \geq 0\), Proposition 6 also implies \(\lim_{n \to \infty} W(h_{\tau + \tau_n}) = \overline{W}\). Note that \(\lim_{n \to \infty} h_{\tau + \tau_n}(t) = \lim_{n \to \infty} h(t + \tau + \tau_n) = h^*(t + \tau) = (h^*)_t(t)\) a.e.. In sum \(W(h^*) = \overline{W} = W((h^*)_t)\) \(\forall \tau\). By Corollary 3 (with \(\tau^* = 0\)), this implies \(h^*\) is constant.

**Proposition 8** If \(h\) is an optimal logging policy then \(\lim_{t \to \infty} h(t)\) and \(\lim_{t \to \infty} t - v(t)\) exist.

Proof: The rotation function \(v\) is monotone increasing so has at most a countable number of jumps. Each jump can be associated with a \(\tau\) where \(h(\tau) = 0\) (see Appendix A, Heaps (2015)). If the number of jumps was infinite, there would be an increasing unbounded \(\{\tau_n\}\) such that \(h(\tau_n) = 0\) \(\forall \tau\). Proposition 7 applies to a subsequence of \(\{\tau_n\}\) which is also impossible since then \(h^* = \lim_{n \to \infty} h(\tau_n) = 0\) which is not an optimal policy. Thus there exists \(T < \infty \ni v\) is continuous on \([T, \infty)\). This implies \(X = \text{cl}(\bigcap_{\tau \in \mathbb{R}^+} X_{\tau} = \{t - v(t) : t \geq \tau\}\) is a nonempty connected interval. Further there is \(x^* \in X\) such that the constant function \(h^* = A/x^*\) is an optimal policy which must satisfy the MUFMP. Note that by (4a) the constant optimal policy \(h^*\) has

\[
W(h^*) = e^{\int_{x^*}^{\infty} [p(x^* - C(h^*)) e^{-rt} dt] + e^{\int_{x^*}^{\infty} h^* e^{-rt} dt] \int_{u=0}^{x^*} E(u)e^{-ru} du dt}
\]
\[ f(x) = p(x)A/x - C(A/x) + e^{rx}(A/x) \int_{u=0}^{x} E(u)e^{-ru} du = r\vec{W} \]

so \( x^* \) is a solution of the following equation.

\[ f(x) = p(x)A/x - C(A/x) + e^{rx}(A/x) \int_{u=0}^{x} E(u)e^{-ru} du = r\vec{W} \]

A connected interval of solutions of this equation can only have nonempty interior if \( f' \equiv 0 \) on such an interval. Here \( f' \) has the sign of \( x\hat{p}(x) - p(x) + C'(h) + xE(x) + (rx-1)e^{rx} \int_{u=0}^{x} E(u)e^{-ru} du \). Now since \( h^* \) is an optimal policy it must satisfy the EXFMP. Maximization of the Hamiltonian implies

\[ p(x^*) - C'(h^*) + \int_{u=0}^{x^*} E(u)e^{-ru} du = q^*(t) \]

a constant. Also the adjoint equation (9) must be satisfied so

\[ rq^* = (\hat{p}(x^*) + E(x^*))(1 - e^{-rx^*}) \]

Combining these two equations then gives a modified Hartman formula\(^{17}\)

\[ r(p(x) + \int_{u=0}^{x} E(u)e^{-ru} du) - (\hat{p}(x) + E(x))(1 - e^{-rx}) = rC'(h) \]

(11)

Using this it can be seen that \( f'(x) \) has the sign of \( (rx-1 + e^{-rx})\hat{p}(x) + E(x) + re^{rx} \int_{u=0}^{x} E(u)e^{-ru} du \) at \( x = x^* \) which is positive. It follows that \( X \) consists of a single point determined by \( A \) and it is this point that is \( \lim_{t \to \infty} t - v(t) \).

In other words, the optimal logging policy will over the long term convert the age distribution of the trees on the forest land into a normal uneven aged forest with the number of age classes given by (11). The solution of this equation is discussed in the appendix. It has a unique solution for all \( A \geq 0 \) where \( A \) is the total area of forest.

7. Conclusion

A long standing conjecture in the management of multiple age class forests has been that optimal harvesting strategies over time change the age distribution of the forest in a way that converges on a normal age distribution. This has now been rigorously proven under simple conditions by Heaps (2015) when only the value of the timber harvested is considered in determining the optimum. This paper considers a version of that model that also includes non-timber values in the determination of the optimum. It shows that an optimal harvesting policy exists for the multiple use model when the present value obtained from the non-timber values is finite when the forest is never harvested \( (\vec{W}_{NT} < \infty) \).

\(^{17}\)The formula is equation (10) in Hartman (1976) when \( C(h) \equiv 0 \).
Moreover, it provides a condition under which the convergence result is valid for this model \((\dot{p}(a) - rp(a) + \dot{E}(a) < 0 \text{ for } a > a_0)\).

An interesting case is a forest for which harvested timber has no market value \((p(a) = 0)\). Then provided logging is not too costly and non-timber values are declining in age for longer ages \((E < 0 \text{ for } a > a_0)\), the optimal age distribution of the forest converges to a normal forest. For other patterns of nontimber values, Proposition 2 says that there is an optimal harvesting policy but this paper does not disclose the asymptotic characteristics of such a policy. Logging a year trees can only be part of an optimal policy if 
\[
-C'(0) + \overline{W}_{NT} > e^{ra} \int_{a}^{\infty} E(u)e^{-ru}du
\]
(see footnote 6). This is not satisfied by any \(a\) if \(E(a) > 0\) \(\forall a\) so the optimal logging policy in this case is no logging at all.\(^{18}\) In the case of a U - shaped non-timber value function, an optimal policy will at most harvest trees from a bounded interval of ages.

**Appendix - The Solution of the Modified Hartman Formula (11)**

\(\underline{a}\) is the solution of \(V(a) = \overline{W}_{NT}\) with \(h = 0\). Thus it satisfies
\[
(p(a) - C'(0))e^{-ra} + \int_{a}^{0} E(u)e^{-ru}du = (1 - e^{-ra})\overline{W}_{NT}
\]
(12)

It also must satisfy \(V'(a) \geq 0\) which is
\[
\dot{p}(a) - r(p(a) - C'(0)) + E(a) - r\overline{W}_{NT} \geq 0
\]
(13)

The modified Hartman formula (11) is
\[
\phi(a) = r(p(a) + \int_{0}^{a} E(u)e^{-ru}du) - (\dot{p}(a) + E(a))(1 - e^{-ra}) = rC'(h)
\]

Solving (12) for the integral and substituting in \(\phi\) shows that
\[
\phi(a) = (r(p(a) + \overline{W}_{NT}) - (\dot{p}(a) + E(a))(1 - e^{-ra}) + rC'(0)e^{-ra}
\]

which by (13) is less than \(rC'(0)\). By condition (f), \(\phi(a)\) is a strictly increasing function of \(a\) for \(a > a_0\). \(C'(h)\) has a minimum value of \(C'(0)\) and by (g) \(\lim_{h \to \infty} C'(h) \geq \overline{p} + \overline{W}_{NT}\) which is \(\geq (1/r) \lim_{a \to -\infty} \phi(a)\). Thus (11) has a unique solution \(h(a)\) for all \(a \geq a_{\text{min}}\) where \(\phi(a_{\text{min}}) = rC'(0)\) and \(h'(a) > 0\). Finally, \(\lim_{a \to -\infty} h(a)a = \infty\) and \(h(a_{\text{min}})a_{\text{min}} = 0\) so \(h(a) = A\) has a unique solution for \(a\) for all values of \(A \geq 0\).

\(^{18}\) \(E(a) > 0 \forall a\) implies \(g(a) = e^{ra} \int_{a}^{\infty} E(u)e^{-ru}du > E(a)/r \forall a\). Thus \(\dot{g}(a) > 0 \forall a\) and \(g(a) > g(0) = \overline{W}_{NT} \forall a > 0\).
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