Textbook Reading: Chapters: 1, 2, Appendix B **Due Date:** Friday, January 22, 2021 by 11:59pm

Instructions

- Upload a copy of your assignment, including a cover page, (pdf format) to the Crowdmark link you've received via email.
- *Correctness, Clarity, & Conciseness* of presentation are reflected in the grading.
- Collaborative discussion on the assignment in encouraged, but the write-up should reflect you own understanding & results. Acknowledge colleagues, TA, or other assistance you received.

Questions

1. Consider the starting arrangement of tiles for the Swap puzzle in Figure 1. In each of the following cases, clearly write out the moves you are making. For example, make a table of the positions where the first row is the starting position and each row is the result of your move applied to the configuration in the previous row. In essence create a record of your solution so anyone reading it will see exactly how you solved the puzzle.

   (a) Solve the puzzle using only legal moves of the form: *swap the contents of any two boxes*.

   (b) Solve the puzzle using only legal moves of the form: *the contents of any box can only be swapped with box* 1.

   (c) Solve the puzzle using only legal moves consisting of 3-cycles: *pick three boxes and cycle the contents either to the right (clockwise) or to the left (counterclockwise)*.

1	2	3	4	5	6
1	6	4	2	3	5

Figure 1: Swap position for Exercise 1.

2. Consider the variation of the Swap puzzle using only legal moves consisting of 3-cycles: *pick three boxes and cycle the contents either to the right (clockwise) or to the left (counterclockwise)*. Show that, starting with the arrangement given in Figure 2, it is possible to put the numbers in each of the orders:
   (a) 123465, and (b) 612345.

   (c) Try to put it in the solved state: 123456. Were you able to? If so, present your list of moves. If not, how close to the solved state could you get?

1	2	3	4	5	6
2	6	4	1	3	5

Figure 2: Swap position for Exercises 2.

3. **(Understanding Set-Builder Notation)** Determine all the elements of the following sets.

   (a) $\{1 + (-1)^n \mid n \in \mathbb{N}\}$

   (b) $\{n \in \mathbb{N} \mid n \leq 25$ and n is divisible by $3\}$

   (c) $\{n \in \mathbb{N} \mid n \leq 25,\ n$ is prime, and $2n + 1$ is divisible by $3\}$

4. Show that integers $a, b, d \in \mathbb{Z}$, if $d \mid a$ and $d \mid b$ then $d \mid (a + b)$.

5. For each of the following pairs of numbers a, b below compute $\gcd(a, b)$ by hand (see Example B.1). In each case use the Extended Euclidean Algorithm to find integers u and v such that $\gcd(a, b) = ua + bv$. (Hint: Follow the example done in the paragraph before Theorem B.1.4.)

   (a) $306, 702$ (b) $888, 3071$

6. If $\gcd(a, b) = d$ then show $\gcd(a/d, b/d) = 1$.

7. Are there integers x and y that satisfy $1034x + 444y = 1$? If yes, determine a solution. If no, give a reason why not.

8. If $d \mid ab$ does it follow that $d \mid a$ or $d \mid b$?

9. Use the Extended Euclidean Algorithm to show the following:

   (a) Let $\gcd(a, b) = d$ and suppose that $c \mid a$ and $c \mid b$, show that $c \mid d$.

   (b) If $d \mid ab$ and $\gcd(d, a) = 1$, show that $d \mid b$.

10. Use Theorem B.3.1 to calculate $\phi(42)$, $\phi(420)$, and $\phi(4200)$.

11. Find all m such that $1848 \equiv 1914 \mod m$.

12. When 3^{2027} is expressed as an integer, what is last digit (i.e. the *ones* digit) of 3^{2027}? What are the last two digits of 3^{2027}?

13. Is $1 + 2 + 2^2 + 2^3 + \cdots + 2^{2020} + 2^{2021}$ divisible by 3? Justify your answer.