# Math 304 Assignment 2 – Solutions

1. (a) $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

$$\alpha\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \varepsilon$$

(b) $\beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 6 & 1 & 4 & 8 & 3 & 7 \end{pmatrix}$

$$\beta\beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 7 & 5 & 2 & 3 & 8 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 6 & 1 & 4 & 8 & 3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = \varepsilon$$

2. $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 3 & 7 & 1 & 8 & 5 & 4 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 6 & 7 & 1 & 3 & 8 & 2 \end{pmatrix}$

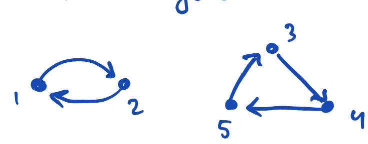
$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 7 & 6 & 5 & 4 & 3 & 2 & 8 \end{pmatrix}$

(a) $$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 3 & 7 & 1 & 8 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 6 & 7 & 1 & 3 & 8 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 8 & 5 & 2 & 1 & 7 \end{pmatrix}$$

(b) $$\alpha\gamma\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 3 & 7 & 1 & 8 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 7 & 6 & 5 & 4 & 3 & 2 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 6 & 7 & 1 & 3 & 8 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 3 & 4 & 5 & 2 & 7 & 1 \end{pmatrix}$$

(c) $$\alpha^{-1}\gamma\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 3 & 8 & 7 & 1 & 4 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 7 & 6 & 5 & 4 & 3 & 2 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 3 & 7 & 1 & 8 & 5 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 5 & 8 & 4 & 2 & 6 & 1 & 3 \end{pmatrix}$$

3. If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$, then it has order 6 since it is a 2-cycle and a 3-cycle:




(a) Therefore, $\alpha^{42} = (\alpha^3)^{14} = \varepsilon^{14} = \varepsilon$.

(b) Since $\alpha^{2021} = \alpha^{2019}\alpha^2 = (\alpha^3)^{673}\alpha^2 = \varepsilon\cdot\alpha^2 = \alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 3 & 4 \end{pmatrix}$ then

$\alpha^{2021} \neq \varepsilon$.

4. Let $m = \mathrm{ord}(\alpha)$ and $n = \mathrm{ord}(\alpha^{-1})$.
Then $\alpha^m = \varepsilon \Rightarrow \alpha^m \cdot \alpha^{-m} = \alpha^{-m} \Rightarrow \varepsilon = \alpha^{-m} \Rightarrow \mathrm{ord}(\alpha^{-1}) \le m.$
$\Rightarrow n \le m$.

On the other hand,
$\alpha^{-n} = \varepsilon \Rightarrow \alpha^{-n}\alpha^n = \alpha^n \Rightarrow \varepsilon = \alpha^n \Rightarrow \mathrm{ord}(\alpha) \le n$
$\Rightarrow m \le n$.

Therefore $m = n$ so $\mathrm{ord}(\alpha) = \mathrm{ord}(\alpha^{-1})$.

5. (a) For $n=0$ the result is trivial (both sides are $\varepsilon$).
For $n>0$,

$$(\alpha^{-1}\beta\alpha)^n = (\alpha^{-1}\beta\alpha)(\alpha^{-1}\beta\alpha)\cdots(\alpha^{-1}\beta\alpha)$$

$$= \alpha^{-1}\beta(\underbrace{\alpha\alpha^{-1}}_{\varepsilon})\beta(\underbrace{\alpha\alpha^{-1}}_{\varepsilon})\beta\cdots(\underbrace{\alpha\alpha^{-1}}_{\varepsilon})\beta\alpha, \text{ by associativity}$$

$$= \alpha^{-1}\beta^n\alpha.$$

Therefore, result is true for positive integers $n$.

For $n<0$,

this is a positive integer

$$(\alpha^{-1}\beta\alpha)^n = \left((\alpha^{-1}\beta\alpha)^{-1}\right)^{-n} = (\alpha^{-1}\beta^{-1}\alpha)^{-n} = \alpha^{-1}(\beta^{-1})^{-n}\alpha, \text{ since } -n>0$$

$$= \alpha^{-1}\beta^n\alpha.$$

Therefore, result is true for all integers $n$.

(b) Let $m = \mathrm{ord}(\beta)$ and $k = \mathrm{ord}(\alpha^{-1}\beta\alpha)$. Then

$$\begin{aligned}(\alpha^{-1}\beta\alpha)^m &= \alpha^{-1}\beta^m\alpha, && \text{by part (a)}\\ &= \alpha^{-1}\varepsilon\alpha, && \text{since } m=\mathrm{ord}(\beta)\\ &= \alpha^{-1}\alpha\\ &= \varepsilon\end{aligned}$$

Therefore, $k \mid m$ by Theorem 3.8.2.

On the other hand,

$$\begin{aligned}\varepsilon &= (\alpha^{-1}\beta\alpha)^k && \text{since } k=\mathrm{ord}(\alpha^{-1}\beta\alpha)\\ &= \alpha^{-1}\beta^k\alpha && \text{by part (a).}\\ &= \beta^k && \text{by cancellation}\end{aligned}$$

Therefore, $m \mid k$ by Theorem 3.8.2.

Since $k \mid m$ and $m \mid k$ then $m = k$. $\square$

6. $\alpha\beta\gamma\beta^{-1}\alpha = \alpha\beta\sigma\beta^{-1}\alpha \iff \gamma\beta^{-1}\alpha = \sigma\beta^{-1}\alpha$ by left cancellation of $\alpha\beta$

$\iff \gamma = \sigma$ by right cancellation of $\beta^{-1}\alpha$.

7. $\alpha = (3\ 7\ 4)$ , $\beta = (5\ 10\ 6)(2\ 9\ 4\ 7)(3\ 8)$

(a) $\alpha\beta = (3\ 7\ 4)(5\ 10\ 6)(2\ 9\ 4\ 7)(3\ 8)$
$= (2\ 9\ 4\ 8\ 3)(5\ 10\ 6)$

(b) $\alpha^{-1} = (3\ 4\ 7)$

(c) $\mathrm{ord}(\alpha) = 3$

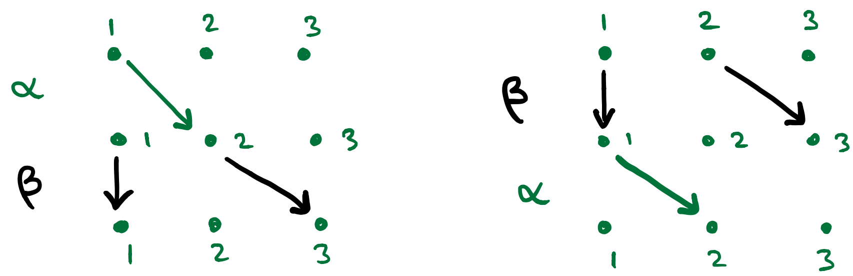
(d) $\mathrm{ord}(\beta) = \mathrm{lcm}(3,4,2) = 12$

8. Let's start with an example, though this isn't necessary in providing a general proof it may provide insight into how to find a general argument.

Consider $\alpha$ such that $\alpha(1) = 2$. (note: I haven't assumed anything else about $\alpha$, and we'll see it doesn't matter.) Then clearly $\alpha(3) \neq 2$ since $\alpha$ is one-to-one. This will be important to remember for later in this argument. We'll create a new function $\beta$ in which

$$(\beta\alpha)(1) \neq (\alpha\beta)(1)$$

from which it will follow that $\alpha\beta \neq \beta\alpha$.




In the diagram above we want $(\alpha\beta)(1)$ to end up with a different value than $(\beta\alpha)(1)$. So we'll define $\beta$ by

$$\beta(1) = 1 \quad , \quad \beta(2) = 3 \qquad (\text{and so } \beta(3) = 2)$$

This means $\beta = (2\ 3)$. We see that $(\alpha\beta)(1) = 3$ and $(\beta\alpha)(1) = 2$. Therefore,

$$\alpha\beta \neq \beta\alpha.$$

General argument : Here we don't make reference to particular numbers since we don't know what numbers $\alpha$ affects. We'll use variables $a, b, c$ in place of $1, 2, 3$.

Proof: Let $\alpha \in S_n$ where $\alpha \neq \varepsilon$. Then there are two numbers $a, b \in [n]$ such that $\alpha(a) = b$. Since $n \geq 3$ there is another number $c \in [n]$ different from $a$ and $b$.
Define $\beta$ to be the 2-cycle

$$\beta = (b\ c)$$

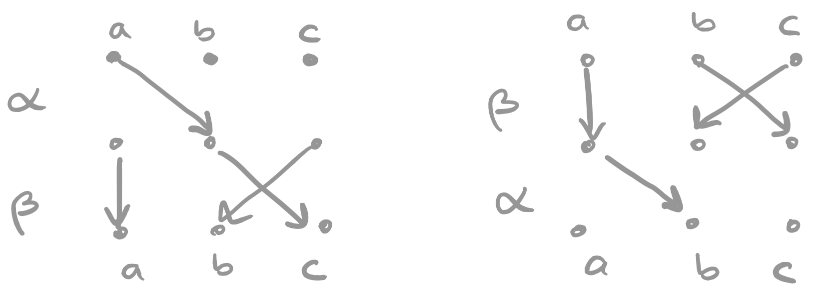
Then

$$(\alpha\beta)(a) = \beta(\alpha(a)) = \beta(b) = c$$

but

$$(\beta\alpha)(a) = \alpha(\beta(a)) = \alpha(a) = b$$

Therefore $\alpha\beta \neq \beta\alpha$. $\square$




$(\alpha\beta)(a) = c$ $\qquad$ $(\beta\alpha)(a) = b$

9. By Theorem 4.4.1 we have order is

$$\mathrm{lcm}(6, 12, 26) = 4 \cdot 3 \cdot 13 = 156$$

10. Suppose $\alpha \in S_5$ has order 7. Then the disjoint cycle form of $\alpha$ must contain at least one cycle of length 7 (since the order is the lcm of cycle lengths, and 7 is prime). But it is impossible to have a 7-cycle on only 5 objects.. Therefore, no such $\alpha$ exists.

11. For $\alpha = (1\ 7\ 4\ 5\ 9)(3\ 8)(10\ 6\ 2)$ we have

$$\alpha^m = (1\,7\,4\,5\,9)^m\,(3\,8)^m\,(10\ 6\ 2)^m.$$

If this is a 3-cycle then

$$(1\ 7\ 4\ 5\ 9)^m = \varepsilon, \quad (3\,8)^m = \varepsilon, \quad \& \quad (10\ 6\ 2)^m \text{ is a 3-cycle}$$

Therefore, $m$ must be divisible by 10 but not divisible by 3.
(So $m$ could be 10 or 20, but not 30.)

12. (a) Consider $\alpha = (1\ 2\ 3)$, $\beta = (1\ 4\ 5)$. Then $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta) = 3$ but

$$\alpha\beta = (123)(145) = (1\ 2\ 3\ 4\ 5)$$

has order 5.

(b) Consider $\alpha = (123)(678)$, $\beta = (145)(679)$. Then $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta) = 3$ but

$$\alpha\beta = (12345)(69)(78)$$

has order 10.

13. If $\beta^3 = \varepsilon$ then $\beta^{-1}$ also satisfies $(\beta^{-1})^3 = \varepsilon$. That is, (non-identity) solutions to $\alpha^3 = \varepsilon$ come in pairs $\{\beta, \beta^{-1}\}$. If $\beta = \beta^{-1}$ then $\beta^2 = \varepsilon$, and $\beta^3 = \varepsilon$, so $\beta = \varepsilon$. Therefore, in only one case do we not get a pair of distinct solutions. We can label the solutions as follows

$$\underbrace{\varepsilon}_{\text{single}},\ \underbrace{\alpha_1, \alpha_1^{-1}}_{\text{pair}},\ \underbrace{\alpha_2, \alpha_2^{-1}}_{\text{pair}},\ \dots$$

Therefore, we have an odd number of solutions.