Math 304 Assignment 3 – Solutions

1. (a) $\alpha = (1\ 5)(2\ 3\ 6\ 7\ 8\ 4)$

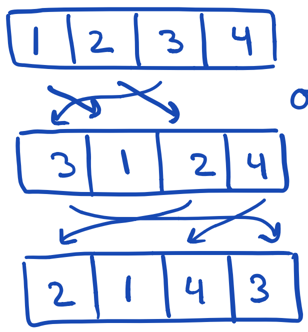
   (b) $\tau_1 = (2\ 3)$, $\tau_2 = (1\ 5)$, $\tau_3 = (7\ 8)$

   (c) $\tau_1\tau_2\tau_3 = (2\ 3)(1\ 5)(7\ 8)$

   (d) $\beta = (3\ 6\ 8\ 4)$

   $$\begin{aligned}\alpha\tau_1\tau_2\tau_3 &= (1\ 5)(2\ 3\ 6\ 7\ 8\ 4)(2\ 3)(1\ 5)(7\ 8)\\ &= (3\ 6\ 8\ 4)\end{aligned}$$

   It follows that $\alpha\tau_1\tau_2\tau_3 = \beta$.

2. 

$\sigma_1 = (1\ 2\ 3)$

$\sigma_2 = (1\ 4\ 3)$

From the puzzle moves on the left

$$\varepsilon\sigma_1\sigma_2 = (1\ 2)(3\ 4)$$

Therefore,

$$\underbrace{(1\ 2\ 3)(1\ 4\ 3)}_{\text{product of 3-cycles}} = (1\ 2)(3\ 4)$$

Alternate solution: Here we use the factorization fact $(a\ b\ c) = (a\ b)(a\ c)$.

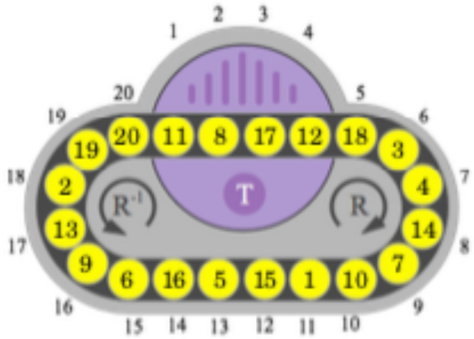
$$\begin{aligned}(1\ 2)(3\ 4) &= \underbrace{(1\ 2)(1\ 3)}\underbrace{(1\ 3)(3\ 4)}, && \text{since } (1\ 3)(1\ 3) = \varepsilon\\ &= (1\ 2\ 3)(3\ 1\ 4) && \text{, since } (1\ 3)(3\ 4) = (3\ 1)(3\ 4) = (3\ 1\ 4)\\ &= (1\ 2\ 3)(1\ 4\ 3) && \text{, since } (3\ 1\ 4) = (1\ 4\ 3).\end{aligned}$$

3. $(1\ 5\ 3\ 12\ 15\ 6\ 2)(7\ 11\ 14\ 8\ 9\ 10\ 16)$

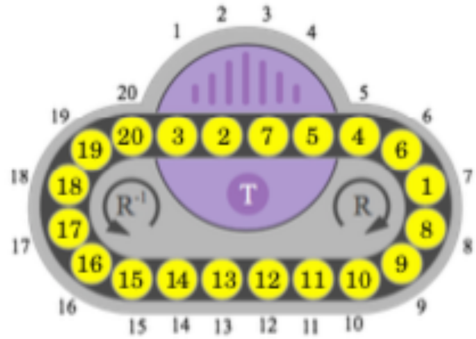
4.

1: 15	2: 8	3: 3	4: 2
5: 10	6: 12	7: 6	8: 8
9: 9	10: 1	11: 11	12: 7
13: 13	14: 14	15: 5	16:

5.



$\xrightarrow{\alpha}$



(a) Starting position:

$\beta = (1\ 11)(2\ 18\ 5\ 13\ 17\ 3\ 6\ 15\ 12\ 4\ 7\ 9\ 16\ 14\ 8)$

(b) $\alpha = (1\ 11\ 7\ 5\ 18\ 2\ 8\ 14\ 16\ 9\ 3\ 17\ 13\ 4\ 12\ 15\ 6)$

(c) $\beta\alpha = (1\ 11)(2\ 18\ 5\ 13\ 17\ 3\ 6\ 15\ 12\ 4\ 7\ 9\ 16\ 14\ 8) \cdot (1\ 11\ 7\ 5\ 18\ 2\ 8\ 14\ 16\ 9\ 3\ 17\ 13\ 4\ 12\ 15\ 6)$

$= (1\ 7\ 3)(4\ 5)$

This is the configuration of puzzle on the right.

6. (a) $(2\ 4\ 7\ 9)(3\ 5\ 8) = (2\ 4)(2\ 7)(2\ 9)(3\ 5)(3\ 8)$

product of 5 2-cycles $\Rightarrow$ odd

(b) $(1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10) = (1\ 2)(1\ 3)(1\ 4)(1\ 5)(6\ 7)(6\ 8)(6\ 9)(6\ 10)$

product of 8 2-cycles $\Rightarrow$ even

7.

1	2	3	4
5	6	7	8
14	10	12	15
9	13	11	

(a)

13	9	5	3
2	1	7	10
	15	14	8
12	11	6	4

(b)

permutation: $(9\ 13\ 14)(11\ 15\ 12)$

$= (9\ 13)(9\ 14)(11\ 15)(11\ 12)$

product of 4 2-cycles

$\therefore$ permutation is even

permutation: $(1\ 6\ 15\ 10\ 8\ 12\ 13)(2\ 5\ 3\ 4\ 16\ 9)(11\ 14)$

Can be expressed as 12 2-cycles.

$\therefore$ permutation is even.

8. Let $\alpha, \beta \in S_n$. Write each as a product of 2-cycles:

$$\alpha = \tau_1 \tau_2 \cdots \tau_k \quad , \quad \beta = \sigma_1 \sigma_2 \cdots \sigma_\ell .$$

the the product $\alpha\beta$ can be written in terms of $k+\ell$ 2-cycles!

$$\alpha\beta = \tau_1 \tau_2 \cdots \tau_k \sigma_1 \sigma_2 \cdots \sigma_\ell .$$

(a) α, β even $\Rightarrow$ k, ℓ even $\Rightarrow$ $k+\ell$ even $\Rightarrow$ $\alpha\beta$ even

(b) α, β odd $\Rightarrow$ k, ℓ odd $\Rightarrow$ $k+\ell$ even $\Rightarrow$ $\alpha\beta$ even

(c) α odd, β even $\Rightarrow$ k odd, ℓ even $\Rightarrow$ $k+\ell$ odd $\Rightarrow$ $\alpha\beta$ odd

Similarly, α even, β odd $\Rightarrow$ $\alpha\beta$ odd. $\square$

9. (a)(b) Let $\alpha = \tau_1 \tau_2 \cdots \tau_k$, where τ_i is a 2-cycle. Then $\alpha^{-1} = \tau_k \cdots \tau_2 \tau_1$, so α and α^{-1} can be expressed using the same number of 2-cycles. Therefore, α and α^{-1} have the same parity.

10. Let $\alpha = \tau_1 \tau_2 \cdots \tau_k$ and $\beta = \sigma_1 \sigma_2 \cdots \sigma_m$ where τ_i, σ_j are 2-cycles. Then

$$\beta^{-1} \alpha \beta = \sigma_m \cdots \sigma_2 \sigma_1 \tau_1 \tau_2 \cdots \tau_k \sigma_1 \sigma_2 \cdots \sigma_m$$

is a product of $m+k+m = k+2m$ 2-cycles. Since $k+2m$ and k have the same parity (both are odd or both are even) then α and $\beta^{-1}\alpha\beta$ have the same parity. $\square$

11. (a) $(1\,2)(3\,4)$ is an even permutation with an even order (it has order 2)

(b) $(1\,2)$ is an odd permutation with an even order. (it has order 2)

(c) Let α be a permutation with odd order. Writing α in disjoint cycle form:

$$\alpha = \sigma_1 \sigma_2 \cdots \sigma_k \quad , \quad \text{where } \sigma_i \text{ is an } \ell_i\text{-cycle}.$$

It follows that ℓ_i must be odd for each i since

$$\mathrm{ord}(\alpha) = \mathrm{lcm}(\ell_1, \ell_2, \ldots, \ell_k) \quad \text{is odd}.$$

Now, a cycle of odd length is an even permutation, and the product of even permutations is even. Therefore, α is an even permutation. $\square$