Math 304 Assignment 4 – Solutions

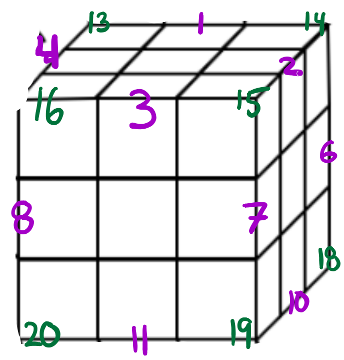
1. For the oval track puzzle with 21 disks the basic moves are

$$R = (21\text{-cycle}) \implies R, R^{-1} \text{ even}$$
$$T = (1\,4)(2\,3) \implies T, T^{-1} \text{ even}$$

Therefore any soluble configuration is a product of R, R^{-1}, T, T^{-1} which is even. Hence, odd permutations are not soluble.
Since the configuration shown is $(1\,3)$ which is odd, it is not soluble.

2 (a) Label the 12 edge cubies 1–12, and label the 8 corner cubies 13–20. Then each of the six cube moves R, L, U, D, F, B is a product of two 4-cycles.
For example,

$$R = \underbrace{(2\ 6\ 10\ 7)}_{\text{edges}}\underbrace{(14\ 18\ 19\ 15)}_{\text{corners}}.$$

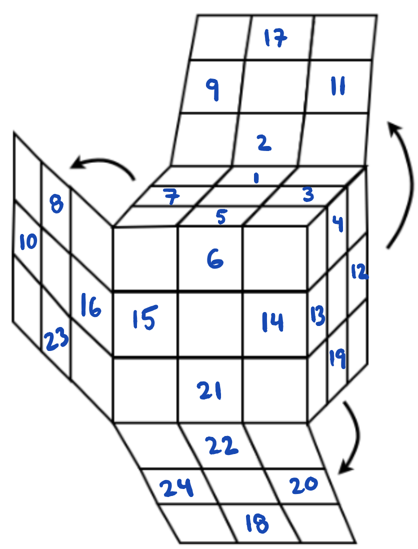


Therefore, each move is an even permutation of cubies, so any soluble position must be an even permutation.

Since swapping two corners is a 2-cycle, which is odd, it is impossible to perform.

(b) Label the 24 edge stickers from 1 to 24. We'll ignore corner cubies. Each of the six cube moves R, L, U, D, F, B is a permutation on these 24 numbers. For example,

$$R = (3\ 11\ 20\ 14)(4\ 12\ 19\ 13)$$



Similarly for the other 5 moves. Each move is therefore an even permutation (two 4-cycles) of stickers. It follows that any product/inverse of them is also even. Hence, only even permutations of the edge stickers are possible.

Since flipping a single edge is a 2-cycle, which is odd, it is impossible to perform.

3. (a) $(1\,2\,3\,4\,5\,6)(7\,8) \in A_9$ contains a 6-cycle.

(b) $(1\,2\,3)(4\,5\,6\,7)(8\,9) \in A_{10}$ contains a 3-cycle and a 4-cycle.

4. $$\sigma = (1\,2\,5\,8)(2\,5\,4\,7)(4\,5\,6)(1\,4)(8\,9)$$
$$= \underbrace{(1\,2)(1\,5)}_{=(1\,2\,5)}\underbrace{(1\,8)(2\,5)}_{=(1\,8\,5)(1\,2\,5)}\underbrace{(2\,4)(2\,7)}_{=(2\,4\,7)}(4\,5\,6)\underbrace{(1\,4)(8\,9)}_{=(1\,4\,9)(1\,8\,9)}$$
$$= (1\,2\,5)(1\,8\,5)(1\,2\,5)(2\,4\,7)(4\,5\,6)(1\,4\,9)(1\,8\,9)$$

Therefore, σ can be written as a product of 3-cycles.

5. (a) Let $\alpha \in S_n$ be an odd permutation. Then $(1\,2)\alpha$ is an even permutation, so by Parity Theorem it can be written as a product of 3-cycles:
$$(1\,2)\alpha = \sigma_1\sigma_2\cdots\sigma_k \quad , \text{ where } \sigma_i \text{ are 3-cycles.}$$
Therefore,
$$\alpha = (1\,2)\sigma_1\sigma_2\cdots\sigma_k$$
is such an expression.

(b) $$\beta = (1\,2\,3)(4\,5\,6\,7)(8\,9\,10)$$
$$= (1\,2)\underbrace{(1\,3)(4\,5)}_{=(1\,3\,5)(1\,4\,5)}\underbrace{(4\,6)(4\,7)}_{=(4\,6\,7)}(8\,9\,10)$$
$$= (1\,2)(1\,3\,5)(1\,4\,5)(4\,6\,7)(8\,9\,10)$$
$\rightsquigarrow$ which is a 2-cycle followed by four 3-cycles.

6. The order of a permutation written in disjoint cycle form is the least common multiple of its cycle lengths, so an element has order 18 if its disjoint cycle form contains a cycle of length divisible by 9 and a cycle length divisible by 2. This means there must be at least $9+2 = 11$ distinct numbers in the cycles. But since we are in A_{10} this is impossible.

A_{10} has no element of order 18.

7. An element of order 5 in A_5 must be a 5-cycle:

$$(\_\ \_\ \_\ \_\ \_).$$

There are $5!$ ways to arrange the numbers 1,2,3,4,5 to make a 5-cycle, however each 5-cycle a 5 different representations:

$$(abcde) = (bcdea) = (cdeab) = (deabc) = (eabcd)$$

Therefore, there are $5!/5 = 4! = 24$ distinct 5-cycles. Hence there are 24 elements of order 5.

An element of order 3 in A_5 must be a 3-cycle:

$$(\_\ \_\ \_).$$

There are $5 \cdot 4 \cdot 3 = 60$ ways to make a 3-cycle, but each 3-cycle has 3 representations: $(a\,b\,c) = (b\,c\,a) = (c\,a\,b)$. Therefore, there are

$$60/3 = 20$$

distinct 3-cycles, and hence 20 permutations of order 3.

An element of order 2 in A_5 must be a product of two 2-cycles:

$$(\_\ \_)(\_\ \_).$$

There are $\binom{5}{2}\binom{3}{2}$ ways to pick the numbers to fill the slots, but disjoint cycles commute so we've overcounted by a factor of 2. Hence there are

$$\frac{\binom{5}{2}\binom{3}{2}}{2} = \frac{\frac{5!}{3!\,2!} \cdot \frac{3!}{2!}}{2} = \frac{5 \cdot 4 \cdot 3}{4} = 15$$

elements of order 2.

8. Let β be an odd permutation in S_n. Then $\beta\gamma^{-1} \in A_n \subset B$ so $\beta = (\beta\gamma^{-1})\gamma \in B$, since $\gamma \in B$ and B closed under multiplication.

Therefore, $O_n \subset B$, thus $B = S_n$.

9. SageMath Explorations :

Sample Data :

a	b	b^(-1) a b
(1,2,3,4)	(1,18,9,11,4,15,16)(2,20)(6,8)(7,14,13)(10,17,19,12)	(3,15,18,20)
(1,2,3,4)	(1,7,5,8)(2,18,4,20,11)(3,9,15)(6,19,10,14,13)(12,16,17)	(7,18,9,20)
(1,2,3,4)	(1,15,3,2,19,4,17,7,18,11,10,9,16,14,5,8)(6,13,20,12)	(2,17,15,19)
(1,2,3,4)	(1,14,6,19,9,17,11,12,15,4,7)(2,18,5,20,10,3,8,16)	(7,14,18,8)
(1,2,3,4)	(1,6,7,17,3,9,18,11,10,12,14,13,20,19,4)(2,15,5,16)	(1,6,15,9)
(1,2,3,4)(5,6)(7,8,9)	(1,14,18,15,19,10,9,11,8,2)(3,17,5,12,4,20,13,6,7,16)	(1,17,20,14)(2,11,16)(7,12)
(1,2,3,4)(5,6)(7,8,9)	(1,10,7,19,5,12,11,18,8,20)(2,9,15,3,6,16,14,17,4,13)	(6,13,10,9)(12,16)(15,19,20)
(1,2,3,4)(5,6)(7,8,9)	(1,4,12,8,10,14,6,19,13,2,7,15,9)(3,16)(5,20,11)	(1,15,10)(4,7,16,12)(19,20)
(1,2,3,4)(5,6)(7,8,9)	(1,8,20,11,14,10,18)(3,13,16,5,7,4)(6,12)(17,19)	(2,13,3,8)(4,20,9)(7,12)
(1,2,3,4)(5,6)(7,8,9)	(1,8,9,10,18,14,11,2,16,6)(3,4,7,5,15)(12,13)(17,20)	(1,15)(4,7,8,16)(5,9,10)
(1,2,3,4)(5,6)(7,8,9)	(1,13,15,17,10,7,5,9,2,16,8,6,14,18)(3,11,12)(19,20)	(2,5,6)(4,13,16,11)(9,14)
(1,2,3,4)(5,6)(7,8,9)	(1,19,4,9,15,17,18,10,16,13,5,8,2,20,12,11)(3,14,6,7)	(2,15,3)(7,8)(9,19,20,14)
(1,2,3,4)(5,6)(7,8,9)	(1,2,4,12,5,13,17,15,3,19,6,7,20,9,14,18,11)(8,16)	(2,4,19,12)(7,13)(14,20,16)
(1,2,3,4)(5,6)(7,8,9)	(1,17)(2,15,7,8,6,11,4,3,16,5)(10,12,19,14)(18,20)	(2,11)(3,17,15,16)(6,9,8)
(1,2,3,4)(5,6)(7,8,9)	(1,14,12,11,8,10,4,15,7,13,19,5,18,6,2,3,20,16,9)	(1,13,10)(2,18)(3,20,15,14)

Consider the first row :

$$\begin{array}{rcl} \alpha & = & (1 \quad 2 \quad 3 \quad 4) \\ & & \ \ \downarrow_\beta \ \downarrow_\beta \ \downarrow_\beta \ \downarrow_\beta \\ \beta^{-1}\alpha\beta & = & (\beta(1) \ \ \beta(2) \ \ \beta(3) \ \ \beta(4)) \end{array}$$

Seems like β gets applied to each entry in the cycle

$$= (18 \ \ 20 \ \ 3 \ \ 15)$$

$$= (3 \ \ 15 \ \ 18 \ \ 20)$$

Check this is true for other entries in the table.

Observations : ① α and $\beta^{-1}\alpha\beta$ have the same cycle structure. For example, if α is the product of two 3-cycles, four 5-cycles, and an 11-cycle, then so is $\beta^{-1}\alpha\beta$. However, the individual numbers in the cycles may be different.

② If α has a cycle

$$(a_1 \ a_2 \ \cdots \ a_k)$$

in its decomposition, then the corresponding cycle in $\beta^{-1}\alpha\beta$ is :

$$(\beta(a_1) \ \ \beta(a_2) \ \cdots \ \beta(a_k))$$

(This is Lemma 14.1.1 on page 176.)