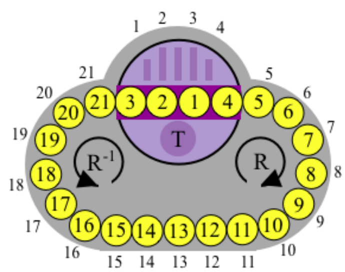
Textbook Reading: Chapters 7, 8 **Due Date:** Friday, February 12, 2021 by 11:59pm

Instructions

- Upload a copy of your assignment (pdf format) to the Crowdmark link you've received via email.
- *Correctness, Clarity, & Conciseness* of presentation are reflected in the grading.
- Collaborative discussion on the assignment in encouraged, but the write-up should reflect you own understanding & results. Acknowledge colleagues, TA, or other assistance you received.

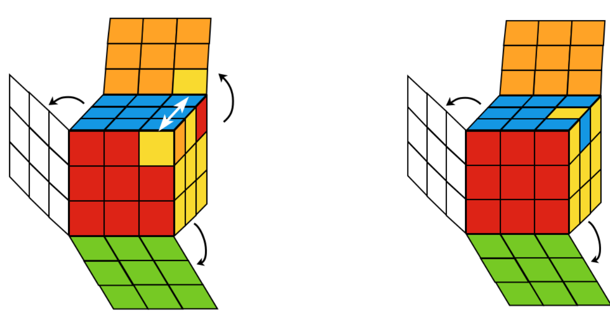
Questions

1. **Transposition on a Variation of Oval Track.** Consider a variation of the Oval Track puzzle where there are now 21 disks instead of 20. The diagram below shows a configuration in which the tiles in positions 1 and 3 have been swapped. Show it is impossible to solve this configuration.
Hint: think parity.



2. **Impossible move on Rubik's** $3 \times 3 \times 3$ **cube**
   (a) Show that it is impossible to find a move sequence that **swaps exactly two corner cubies** of the Rubik's cube, while leaving every other cubie in its home location(that is, in its home location with proper orientation). See Figure 1a.
   (b) Show that it is impossible to find a move sequence that **flips exactly one edge cubie** of the Rubik's cube, while leaving every other cubie in its *home position* (that is, in its home location with proper orientation). See Figure 1b.

   Hint: Label the pieces on the cube in some way so that you can assign to each move a permutation of the labelled pieces. Then use a parity argument to show such configurations are impossible.



(a) corner swap (b) single edges flip

Figure 1: Exercise 2 - Show these configurations are unsolvable

3. (a) Give an example of an element in A_9 which contains a 6-cycle in its disjoint cycle form.
   (b) Give an example of an element in A_{10} which contains at least one 3-cycle, and at least one 4-cycle in its disjoint cycle form.

4. Demonstrate the truth of Theorem 8.3.1 by expressing the even permutation σ as a product of 3-cycles.

$$\sigma = (1\ 2\ 5\ 8)(2\ 5\ 4\ 7)(4\ 5\ 6)(1\ 4)(8\ 9)$$

5. **Expressing odd permutations in terms of 3-cycles and one transposition.**
   (a) Show that all odd permutations in S_n can be expressed using exactly one transposition together with zero or more 3-cycles.
   (b) Demonstrate the truth of this claim by expressing this odd permutation with a single transposition and 3-cycles: $\beta = (1\ 2\ 3)(4\ 5\ 6\ 7)(8\ 9\ 10)$.

6. Show that A_{10} contains no element of order 18.

7. Show that A_5 has 24 elements of order 5, 20 elements of order 3, and 15 elements of order 2.

8. Let $B \subset S_n$ be a set of permutations such that
   - B is closed under multiplication (i.e. if $\alpha, \beta \in B$ then so is $\alpha\beta$), and
   - $A_n \subset B$, and
   - B contains an odd permutation γ.

   Show that $B = S_n$. In words this says that a set of permutations B that is closed under multiplication and contains every even permutation and at least one odd permutation must contain *every* permutation.

9. **Explorations in SageMath:**
   Motivation: If you've played with the 15-puzzle, Rubik's cube, or any of the other permutation puzzle, you will find yourself doing move sequences like: "do a move, do another move, then undo the first move". Notationally, if M_1 denotes the first move and M_2 denotes the second move then the move sequence can be written as: $M_1 M_2 M_1^{-1}$. This exercise begins our investigations to try to uncover why move sequences of this form are useful.

   Objectives: In this question you are to investigate the structure of permutations of the form $\beta^{-1}\alpha\beta$. Questions that you should be thinking about as you experiment are:

   (i) What are the similarities between α and $\beta^{-1}\alpha\beta$?

   (ii) What are the differences between α and $\beta^{-1}\alpha\beta$?

   (ii) Can you find a rule for determining $\beta^{-1}\alpha\beta$ for any α and β, that doesn't actually involve multiplying the permutations.

   Method: To begin your explorations, pick a permutation α. For example, $\alpha = (1\ 2\ 3\ 4)$. Then pick a random element β. This can be done using the "random element generator" in SageMath (see example below). Compute $\beta^{-1}\alpha\beta$. Here is the SageMath code.

```
In [1]:  S=SymmetricGroup(20)    # here I've taken n=20, you can try larger values.
         a=S("(1,2,3,4)")
         b=S.random_element()    # generates a random element every time it is executed
         b                       # shows permutation b in output window

Out[1]:  (1,15)(2,8,9,19,3,13)(4,11,7)(5,10,18,14,12)

In [2]:  b^(-1)*a*b        # compute the product

Out[2]:  (8,13,11,15)
```

   For various choices of α, try a number of random permutations β. Record your results in a table like the one below. As an example, I have done the first line already based on the previous computation. You do not need to type this table, just make your handwriting legible.

α	β	$\beta^{-1}\alpha\beta$
(1 2 3 4)	(1 15)(2 8 9 19 3 13)(4 11 7)(5 10 18 14 12)	(8 13 11 15)

Observations and Results: Based on the data collected in your table, write out your observations. The three questions listed under "Motivation" are a starting point. When you make a conjecture, test it by doing some more computations. Further computations will either add credibility to your conjecture, or will reveal that it is incorrect. If it is incorrect, make another conjecture and then test it through more computations.

I don't want you to hand-in SageMath code. Just show some sample data you collected, and state your observations and results. When making a conjecture about what you are observing, state "**Conjecture:**" followed by a statement which you believe to be correct based on your observations/calculations.