Math 304 Assignment 5 - Solutions

1 (a)

5	1	2	3
9	6	8	4
13	10	7	12
14	15	11	

position permutation:
$(1\ 2\ 3\ 4\ 8\ 7\ 11\ 15\ 14\ 13\ 9\ 5)$
$\Rightarrow$ ODD $\neq$

parity of box with empty space: EVEN

$\therefore$ Not soluble

(b)

1	2	3	4
	9	11	6
14	13	15	8
5	12	10	7

position permutation:
$(5\ 13\ 10\ 15\ 11\ 7\ 16)(6\ 8\ 12\ 14\ 9)$
$\Rightarrow$ EVEN $\neq$

parity of box with empty space: ODD
(5 moves to box 16)

$\therefore$ Not soluble

(c)

8	6	3	
2	7	5	14
9	1	15	12
10	13	11	4

position permutation:
$(1\ 10\ 13\ 14\ 8)(2\ 5\ 7\ 6)$
$(4\ 16)(11\ 15)$
$\Rightarrow$ ODD $=$

parity of box with empty space: ODD
(3 moves to box 16)

$\therefore$ Soluble

2. (a)

1	2	3	4
5	6	7	8
13	9	11	12
10	14	15	

A solution sequence:

r r (r d l u) l l

(b)

11	2	3	4
5	1	7	8
9	10	6	12
13	14	15	

A solution sequence:

$$\underbrace{\text{rrdddrulldlu}}_{\beta}\,\underbrace{(\text{ruld})}_{\text{3-cycle}}\ \beta^{-1}$$

$= \text{rrdddrulldlu(ruld)drurdluuull}$

(c)

1	2	3	4
5	6	7	8
13	10	12	11
9	14	15	

Strategy: Solve by doing $(9\ 11\ 13)(11\ 13\ 12) = (9\ 13)(11\ 12)$

A solution sequence:

① $(9\ 11\ 13)$: first we cycle contents of boxes 9, 11, 13

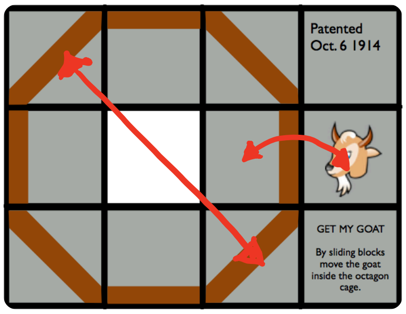
$$\underbrace{(\text{rrdlur})}_{\text{bring tiles 9,13,12 together}}\ \underbrace{\text{drul}}_{\text{cycle them}}\ \underbrace{(\text{ldrull})}_{\text{undo set-up move}}$$

② $(11\ 13\ 12)$: now cycle contents of boxes 11, 13, 12:

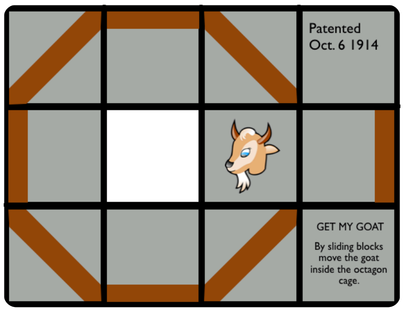
(rrrdlllur)(rdlu)(ldrrrulll)

$\therefore$ full sequence: (rrdlur)(drul)(ldrull)(rrrdlllur)(rdlu)(ldrrrulll)

3. At first glance it may seem like a 2-cycle is required to solve the puzzle, after all we need to get the goat past the gate to its left. However, we know odd permutations of the pieces are impossible (if empty space is to return to where it started) so how can the puzzle be solvable?



Well, any 3-cycle is possible to do (a theorem similar to Thm 9.1 holds for this puzzle too) so we could perform two 2-cycles: the two pieces we want to swap, and another two identical pieces could be swapped. For example, the upper left and lower right fence corners.



The result of this double swap is shown in the figure on the left. Sliding the goat and fence piece to the left captures the goat inside the fenced area.

4. We will show we can find any 3-cycle by using the 2-by-2 array of boxes 9, 10, 13, 14.

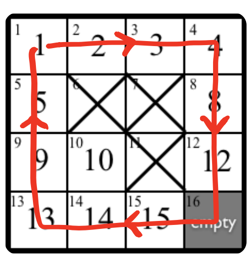
Let a, b, c be any three boxes, and for simplicity lets suppose the tiles in these boxes are a, b, c.

<u>Step 1</u>: Move tile from box a to box 10.

If tile a is not already in box 10, then bring it there by cycling it around the outside track, putting it in box 9, with a space in box 10, then slide it into place.

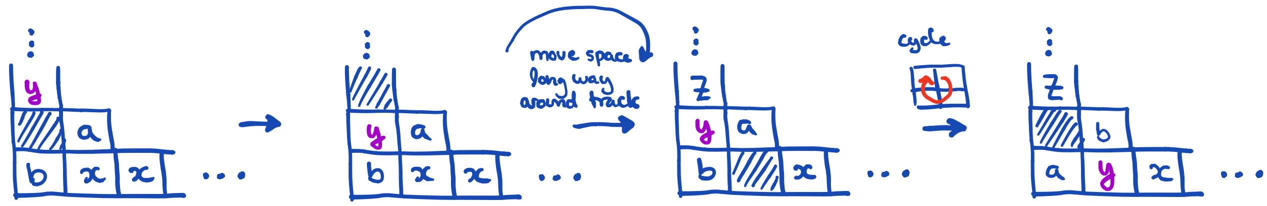
<u>Step 2</u>: Move tile from box b to box 13, and space to box 9.

To do this use the outside track (path shown in red in diagram below) and don't touch box 10.



Step 3: Move tile from box c to box 9, and space to box 14.

This may seem tricky to do, but the point is that we can push any tile through the 2×2 array while leaving tiles a,b in the bottom left corner. This can be done as follows



We've now moved y past a and b. Keep doing this (pushing tiles past tiles a and b) until we move the tile from box c into the 2×2 array.

Let α be the composition of moves used in steps 1 through 3. We now have either

cycle contents so tile $a \mapsto b$, tile $b \mapsto c$, and tile $c \mapsto a$. Call this move σ.

Now undo α and we get $\alpha\sigma\alpha^{-1} = (a\ b\ c)$.

We can therefore produce any even permutation, this proves that Theorem 9.1.1 holds for this version of the 15 puzzle too. Thm 9.1.2 follows from 9.1.1 as it did before.