Math 304 Assignment 7 - Solutions

1 (a) If α can be expressed using k 2-cycles: $\alpha = \tau_1\tau_2\cdots\tau_k$
and β can be expressed using ℓ 2-cycles: $\beta = \sigma_1\sigma_2\cdots\sigma_\ell$
then

$$[\alpha,\beta] = \alpha\beta\alpha^{-1}\beta^{-1} = \tau_1\tau_2\cdots\tau_k\sigma_1\sigma_2\cdots\sigma_\ell\tau_k\cdots\tau_2\tau_1\sigma_\ell\cdots\sigma_2\sigma_1$$

uses $k+\ell+k+\ell = 2(k+\ell)$ 2-cycles, and is therefore even.

(b) $[g,h]=e \Rightarrow ghg^{-1}h^{-1}=e$

$\Rightarrow ghg^{-1}=h$

$\Rightarrow gh = hg$

$\Rightarrow$ g & h commute.

(c) $[g,h]^{-1} = (ghg^{-1}h^{-1})^{-1} = (h^{-1})^{-1}(g^{-1})^{-1}h^{-1}g^{-1} = hgh^{-1}g^{-1} = [h,g]$.

2. Recall that

$$\mathrm{mov}([\alpha,\beta]) \subset \mathrm{mov}(\alpha,\beta) \cup \alpha^{-1}\mathrm{mov}(\alpha,\beta) \cup \beta^{-1}\mathrm{mov}(\alpha,\beta)$$

where $\mathrm{mov}(\alpha,\beta) = \mathrm{mov}(\alpha)\cap\mathrm{mov}(\beta)$. Moreover, since α,β are injective then

$$|\mathrm{mov}(\alpha,\beta)| = |\alpha^{-1}\mathrm{mov}(\alpha,\beta)| = |\beta^{-1}\mathrm{mov}(\alpha,\beta)|.$$

Therefore,

$$|\mathrm{mov}([\alpha,\beta])| \leq 3\,|\mathrm{mov}(\alpha,\beta)|.$$

Using this it now follows that

(a) $|\mathrm{mov}(\alpha)\cap\mathrm{mov}(\beta)| = 0 \Rightarrow |\mathrm{mov}([\alpha,\beta])| = 0 \Rightarrow [\alpha,\beta]=\varepsilon$.

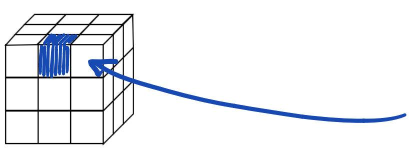
(b) $|\mathrm{mov}(\alpha)\cap\mathrm{mov}(\beta)| = 2 \Rightarrow |\mathrm{mov}([\alpha,\beta])| \leq 6$.

3. Nothing to hand in.

4. Flip 2 adjacent edges.

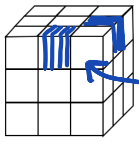
(a) perform the move to verify for yourself

$$\alpha = M_R^{-1} D M_R D^{-1} M_R^{-1} D^2 M_R$$



flips this edge

(b) Take $\beta = U$ so $mov(\alpha,\beta)$ consists of the uf edge. α doesn't bring anything new to this cobicle, β brings the edge from the ur cobicle so the commutator $[\alpha,\beta]$ only affects the uf and ur cobicles.



$[\alpha,\beta]$ flips the two edges as indicated

(c) Verify physically, nothing to write down here.

5. Flip 2 opposite edges:

Could take $\beta = U^2$ in exercise 4 and get a commutator $[\alpha,\beta]$ that does the job. Or take a conjugate of the commutator from 5:

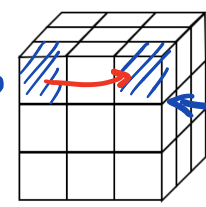
$$B^{-1}R^{-1}\left(\begin{array}{c}\text{apply commutator}\\ \text{from Exercise 4}\end{array}\right)RB.$$

6. Building a corner 3-cycle:

(a) Apply the move sequence ULU^{-1} to a physical cube to see this.

The ufl cube is brought to the rfu position, all other pieces in the right face remain in their original positions.

ufl



rfu

(b) All you need to do is verify (physically), not much to write down here. However, we could see what must happen without doing the moves. Since

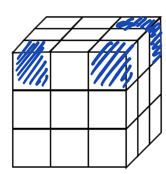
$$mov([\alpha,\beta]) \subset mov(\alpha,\beta) \cup \alpha^{-1}mov(\alpha,\beta) \cup \beta^{-1}mov(\alpha,\beta)$$

then for $\alpha = ULU^{-1}$, $\beta = R^{-1}$ we have

$$mov(\alpha,\beta) = \{rfu\}$$ (look at the cube to see what α and β move)

and α brings ufl into $mov(\alpha,\beta)$, whereas β brings urb into $mov(\alpha,\beta)$. Thus

$$mov([\alpha,\beta]) \subset \{rfu, ufl, urb\}$$



In fact we have equality. Moreover,

- $ufl \xrightarrow{ULU^{-1}} rfu \xrightarrow{R^{-1}} rdf \xrightarrow[\text{no change}]{(ULU^{-1})^{-1}} rdf \xrightarrow{R} rfu$
- $rfu \xrightarrow{ULU^{-1}} dlf \xrightarrow[\text{no change}]{R^{-1}} dlf \xrightarrow{(ULU^{-1})^{-1}} rfu \xrightarrow{R} rub$

  no change
- $rub \xrightarrow[\text{no change}]{ULU^{-1}} rub \xrightarrow{R^{-1}} rfu \xrightarrow{(ULU^{-1})^{-1}} ufl \xrightarrow[\text{no change}]{R} ufl$

7. Two permutations are conjugate in S_n if they have the same cycle structure (Lemma 14.1.1 and subsequent paragraph)

(a) α and β have the same cycle structure, both are a product of a 3-cycle and a 7-cycle. Hence α is a conjugate of β.

$$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9\ 10)$$

$$\beta = (2\ 6\ 3\ 7\ 4\ 10\ 9)(1\ 5\ 8)$$

The permutation $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 6 & 3 & 7 & 4 & 10 & 9 & 1 & 5 & 8 \end{pmatrix}$

has the property

$$\begin{aligned} \gamma^{-1}\alpha\gamma &= (\gamma(1)\ \gamma(2)\ \gamma(3)\ \gamma(4)\ \gamma(5)\ \gamma(6)\ \gamma(7))(\gamma(8)\ \gamma(9)\ \gamma(10)) \\ &= (2\ 6\ 3\ 7\ 4\ 10\ 9)(1\ 5\ 8) \\ &= \beta. \end{aligned}$$

(b) α is the product of one 2-cycle and two 3-cycles, and β is the product of two 2-cycles and one 3-cycle., hence they are not conjugate.

8. $$\begin{aligned} [gag^{-1}, gbg^{-1}] &= (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1} \\ &= ga\underbrace{g^{-1}g}_{e}b\underbrace{g^{-1}g}_{e}a^{-1}\underbrace{g^{-1}g}_{e}b^{-1}g^{-1} \quad \text{by associativity} \\ &= gaba^{-1}b^{-1}g^{-1}, \quad \text{since } gg^{-1} = e \\ &= g[a,b]g^{-1}. \end{aligned}$$

9. (a) $(g_1 g)^h = h^{-1} g_1 g_2 h = h^{-1} g_1 h h^{-1} g_2 h$, since $hh^{-1} = e$

$= (h^{-1} g_1 h)(h^{-1} g_2 h)$ by associativity

$= g_1^h g_2^h$

(b) $g^{h_1 h_2} = (h_1 h_2)^{-1} g (h_1 h_2)$

$= h_2^{-1} h_1^{-1} g h_1 h_2$

$= h_2^{-1} (h_1^{-1} g h_1) h_2$ by associativity

$= h_2^{-1} (g^{h_1}) h_2$ since $h_1^{-1} g h_1 = g^{h_1}$

$= (g^{h_1})^{h_2}$ since $h_2^{-1} x h_2 = x^{h_2}$.