Textbook Reading: Chapters 13, 14 **Due Date:** Friday, March 12, 2021 by 11:59pm

Questions

1. **All about commutators.**
   (a) Let $\alpha, \beta \in S_n$. Show that the commutator $[\alpha, \beta]$ is an even permutation.
   (b) Show that if $[g, h] = e$ then g and h commute.
   (c) Show that $[g, h]^{-1} = [h, g]$.

2. Let $\alpha, \beta \in S_n$.
   (a) If mov(α) and mov(β) have no locations (elements) in common (i.e. mov$(\alpha) \cap$ mov$(\beta) = \emptyset$), what is the permutation $[\alpha, \beta]$?
   (b) If mov(α) and mov(β) have two locations (elements) in common (i.e. $|$mov$(\alpha) \cap$ mov$(\beta)| = 2$), what is the largest $|$mov$([\alpha, \beta])|$ can be?

3. Do exercises 14, 15 in Chapter 13 of the textbook. **You do not need to hand anything in for these questions**, I trust you did them. After all, you are in this course because you wanted to learn about the mathematics of the Rubik's cube, so you wouldn't deny yourself a valuable, and fun, learning opportunity ;-) Playing with these exercises should give you some insight into how to construct your own puzzle moves. Try to create some of your own using commutators.

4. **Flip** 2 **adjacent edges**. In this exercise you will construct a move to produce a double edge flip in the *up* layer, as shown in Figure 1b.

(a) Slice move M_R. (b) Double edge flip

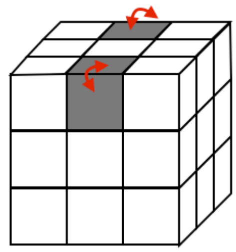
Figure 1: Diagrams for exercise 4

Let M_R denote the "slice move" which consists of rotating the middle slice, parallel to the R face, in the clockwise direction, from the perspective of the R face. See Figure 1a. Consider the move sequence

$$\alpha = M_R^{-1} D M_R D^{-1} M_R^{-1} D^2 M_R.$$

   (a) Verify α flips the edge in the uf position, and fixes everything else in the *up* layer.
   (b) Since α only affects one cubies in the *up* layer, what would be a good choice for move β, so that the commutator $[\alpha, \beta]$ affects only two edge cubies? With your choice of β can you predict the effect of $[\alpha, \beta]$ on the cubies?
   (c) Perform the move $[\alpha, \beta]$ and verify your prediction from the previous part. The move sequence should produce the double edge flip as shown in Figure 1b (or similar).

5. **Flip 2 opposite edges**. Find moves α and β so that the commutator $[\alpha, \beta]$ flips two opposite edges (as shown in the diagram below), and fixes everything else. (Hint: Modify the moves in the previous exercise.)

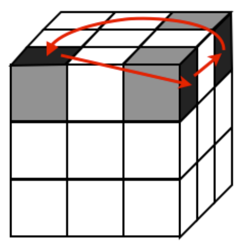


6. **Building a corner 3-cycle.** In this exercise we build a 3-cycle of corners in the *up* layer.

   (a) Verify the the conjugate ULU^{-1} brings one new corner cubie into the *right* face.

   (b) Since ULU^{-1} brings one new corner cubie into the *right* face this makes a good candidate to form a commutator with R^{-1}. Verify the commutator $[ULU^{-1}, R^{-1}]$ moves the corner cubies as indicated in the diagram. The movement of pieces is also given notationally as follows

$$ulf \mapsto ruf, \quad ruf \mapsto rbu, \quad rbu \mapsto ulf.$$



7. For each of the following pairs of permutations state whether they are conjugate in S_{10}. That is, determine whether there exists a $\gamma \in S_{10}$ so that $\beta = \gamma^{-1}\alpha\gamma$. If they are conjugate give an example of a permutation $\gamma \in S_{10}$ so that $\beta = \gamma^{-1}\alpha\gamma$.

   (a) $\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9\ 10)$, $\quad \beta = (1\ 5\ 8)(2\ 6\ 3\ 7\ 4\ 10\ 9)$

   (b) $\alpha = (1\ 5\ 8)(2\ 6)(3\ 7\ 4)$, $\quad \beta = (1\ 2)(7\ 3)(8\ 9\ 10)$

8. Let G be a group. Prove that every conjugate of a commutator is a commutator by showing that $g[a, b]g^{-1} = [gag^{-1}, gbg^{-1}]$ for all $a, b, g \in G$.

9. Show that for $g_1, g_2, h, h_1, h_2 \in G$ the following hold.

   (a) $(g_1g_2)^h = g_1^h g_2^h$

   (b) $g^{h_1h_2} = (g^{h_1})^{h_2}$

   (Recall, the exponential notation g^h is shorthand for conjugation $h^{-1}gh$. These two properties indicate why this shorthand notation is used.)