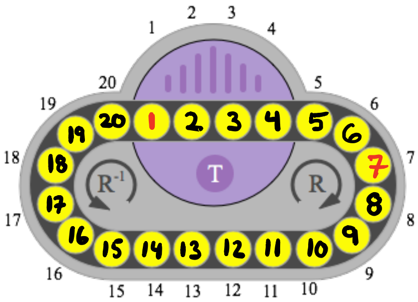
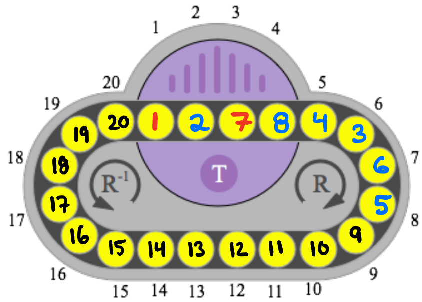
Math 304 Assignment 8 – Solutions

1. First we find a move sequence β^{-1} which takes disk 1 to position 1 and disk 7 to position 3.



$$\beta^{-1} = R^{-4}TR^2TR^2 \longrightarrow$$



Note: We can write β^{-1} as $(3\,6\,7)(4\,5\,8)$, so $\beta = (3\,7\,6)(4\,8\,5)$.

Therefore,

$$\beta^{-1}\sigma_2\beta = \beta^{-1}(1\;3)\beta = \big(\beta(1)\;\beta(3)\big) = (1\;7)$$

2. We want to create the move $(2\;3\;5)$, so we note "2 chases 3".
First we find a move sequence β^{-1} which does the following:

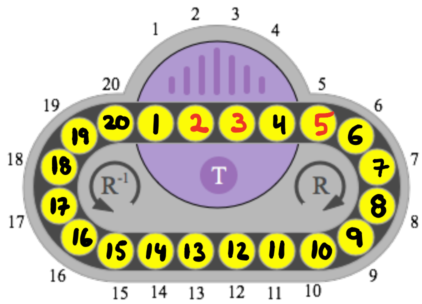
disk 2 $\longrightarrow$ position 1
disk 3 $\longrightarrow$ position 4
disk 5 $\longrightarrow$ position 7

This isn't the only way to do it. You can take disks $\{2,3,5\}$ and put them in positions $\{1,4,7\}$ in any way whatsoever.

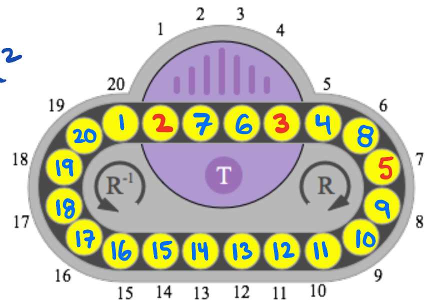
One possible move sequence is

$$\beta^{-1} = R^{-4}TR^2TR^{-1}TR^2$$

This just inserts 2 disks between each pair of disks, then rotates disk 2 to position 1. (Other move sequences can also work to do this)



$$\beta^{-1} = R^{-4}TR^2TR^{-1}TR^2 \longrightarrow$$

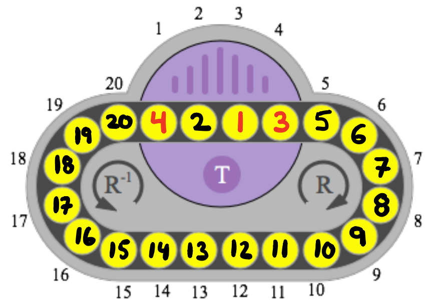


Now disk 2 is in position 1, disk 3 is in position 4, and we want "2 to chase 3", so we need to apply $\sigma_3^{-1} = (1\;4\;7)$.

So

$$\beta^{-1}\sigma_3^{-1}\beta = \big(\beta(1)\;\beta(4)\;\beta(7)\big)$$
$$= (2\;3\;5)$$

3. (a) Configuration : $\alpha = (1\ 3\ 4)$



To solve the puzzle we must perform the permutation
$$\alpha^{-1} = (1\ 4\ 3)$$
which is a 3-cycle.

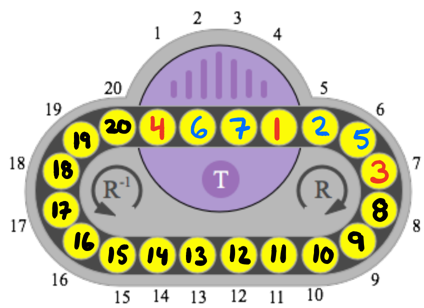
Strategy for solution:

① Move disks in positions 1, 3, 4 to positions 1, 4, 7. That is, move
- disk 4 to position 1 (already done)
- disk 1 to position 4
- disk 3 to position 7

Call the sequence of moves to do this β^{-1}. An example of a move sequence is :
$$\beta^{-1} = R^{-3}TR^{2}TR$$

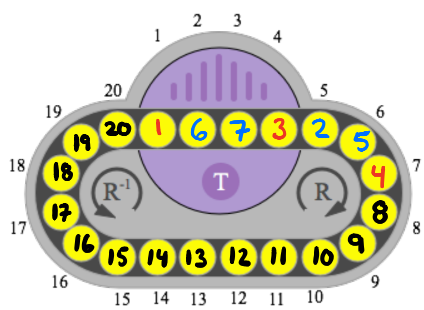
Remember "disk 1 chases disk 4".

$\xrightarrow{\beta^{-1}}$



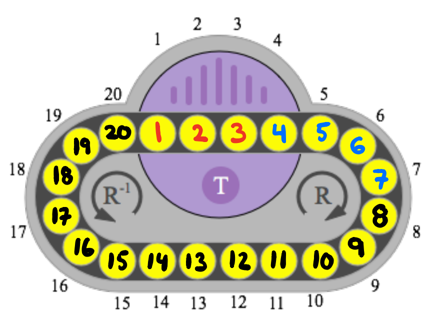
② Apply fundamental 3-cycle $\sigma_3 = (1\ 7\ 4)$:

$\xrightarrow{\sigma_3}$

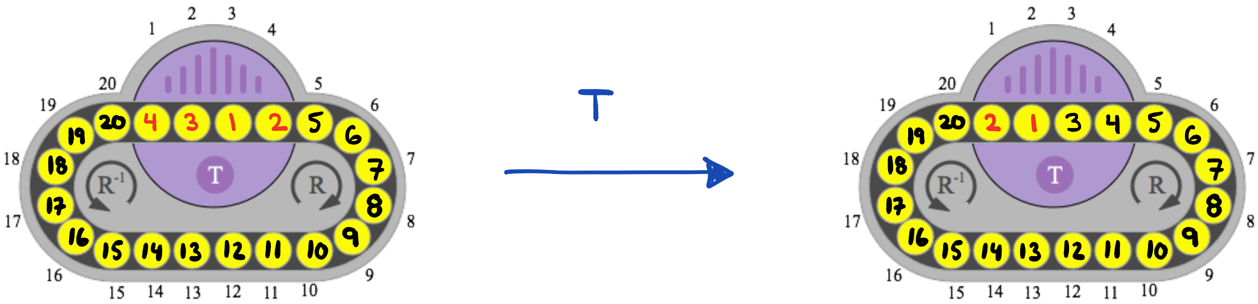


③ Apply β to solve :

$\xrightarrow{\beta}$



3 b) Configuration: $\alpha = (1\,3\,2\,4)$



Now we need to perform a 2-cycle $(1\,2)$. To do this we conjugate the fundamental 2-cycle $\sigma_2 = (1\,3)$. We've already done this in Example 15.2, the move sequence:

$$\beta^{-1} = R^{-1} T R^{-1} T R T R$$

will move the disk from position 2 to position 3, while keeping disk in position 1 in place. Therefore, $\beta^{-1} \sigma_2 \beta$ with solve the puzzle.

Therefore, to solve the puzzle starting from position α we apply

$$T \beta^{-1} \sigma_2 \beta$$

4 (a)

In [3]:
```
# Oval Track - variation T=(1 3 2)
S20=SymmetricGroup(20)
R=S20("(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)")
T=S20("(1,3,2)")
OT=S20.subgroup([R,T])
OT.order() == factorial(20)
```
Out[3]:
```
True
```

$\therefore$ all permutations are possible

In [4]:
```
#Oval Track - variation T=(1 3)(2 4)
S20=SymmetricGroup(20)
R=S20("(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20)")
T=S20("(1,3)(2,4)")
OT=S20.subgroup([R,T])
OT.order() == factorial(20)
```
Out[4]:
```
False
```

$\therefore$ not all permutations are possible.