# Math 304 Assignment 9 - Solutions

1. $g \mathcal{R} h \iff g$ is a conjugate of $h$
$\iff g = k^{-1}hk$ for some $k \in G$.

reflexive: since $g = e^{-1}ge$ where $e \in G$ then $g\mathcal{R}g$.

symmetric: $g\mathcal{R}h \iff g = k^{-1}hk$
$\iff kgk^{-1} = h$
$\iff h = (k^{-1})^{-1}g\,k^{-1}$
$\iff h\mathcal{R}g$

transitive: $g_1\mathcal{R}g_2,\ g_2\mathcal{R}g_3 \Rightarrow g_1 = h^{-1}g_2h,\ g_2 = k^{-1}g_3k$
$\Rightarrow g_1 = h^{-1}(k^{-1}g_3k)h$
$= (kh)^{-1}g_3(kh)$
$\Rightarrow g_1\mathcal{R}g_3$.

Therefore, $\mathcal{R}$ is an equivalence relation.

2. Consider $\mathbb{Z}_{12}$ & $H = \langle 4\rangle = \{0,4,8\}$

(a) (Recall $\sim_H$ is symmetric so order doesn't matter)
(i) $7-3 \equiv 4 \in H \Rightarrow 7 \sim_H 3$
(ii) $11-5 = 6 \notin H \Rightarrow 11 \not\sim_H 5$
(iii) $9-6 = 3 \notin H \Rightarrow 9 \not\sim_H 6$

(b) There are $|\mathbb{Z}_{12}|/|H| = 12/3 = 4$ left cosets

$H = \{0,4,8\}$, $1+H = \{1,5,9\}$
$2+H = \{2,6,10\}$, $3+H = \{3,7,11\}$

3. $\alpha \sim_{A_n} \beta \iff \alpha^{-1}\beta$ is even $\iff \alpha$ & $\beta$ have the same parity

(a) $\left.\begin{array}{ll}\alpha = (23)(46) & \text{even}\\ \beta = (13574) & \text{even}\end{array}\right\} \Rightarrow \quad \therefore\ \alpha \sim_{A_7} \beta$

(b) $\left.\begin{array}{ll}\alpha = (1372) & \text{odd}\\ \beta = (24365) & \text{even}\end{array}\right\} \Rightarrow \quad \therefore\ \alpha \not\sim_{A_7} \beta$

4. (a) The number of left cosets of $H$ in $G$ is $\frac{|G|}{|H|} = 2$.
Since $a \notin H$ then
$H \cap aH = \emptyset$.
Moreover, both have $\frac{1}{2}|G|$ elements, so
$H \cup aH = G$

(b) If $a^nH = a^{n+1}H$ then $a^{-n}a^{n+1} \in H \Rightarrow a \in H$
Therefore, $a \notin H \Rightarrow a^nH \neq a^{n+1}H$.

(c) Let $b \in G$ have odd order, say $\text{ord}(b) = 2k+1$. Towards a contradiction suppose $b \notin H$. Then

$$H \ \& \ bH$$

are the two distinct left cosets of $H$ in $G$, by part (a). By part (b) we have

$$H,\ bH,\ \underset{= H}{b^2H},\ \underset{= bH}{b^3H},\ \ldots,\ \underset{= H}{b^{2k}H},\ \underset{= bH}{b^{2k+1}H}$$

But $b^{2k+1} = e$ so $b^{2k+1}H = H \Rightarrow bH = H \Rightarrow b \in H$. Contradiction.

$\therefore$ $b \in H$. Since $b$ was an arbitrary element of odd order then $H$ must contain all elements of odd order.

5. (a) $(\underset{5}{\_}\ \underset{4}{\_}\ \underset{3}{\_}) \Rightarrow \dfrac{5 \cdot 4 \cdot 3}{3} = 20$ 3-cycles

& 3 ways to arrange to make same cycle

(b) $(\underset{5}{\_}\ \underset{4}{\_}\ \underset{3}{\_}\ \underset{2}{\_}\ \underset{1}{\_}) \Rightarrow \dfrac{5!}{5} = 4! = 24$ 5-cycles

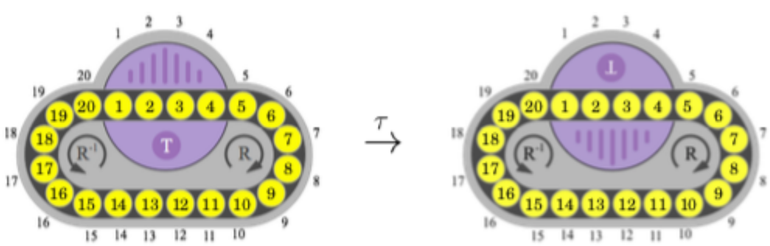
& 5 ways to arrange to make same cycle

(c) Suppose $H < A_5$ of order $30 = \frac{1}{2}|A_5|$ then by question 4 $H$ contains all elements of odd order. In particular $H$ must contain 20 3-cycles, and 24 5-cycles. But this means

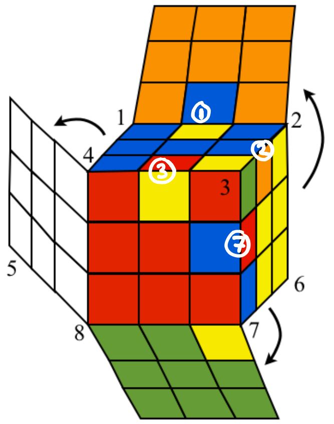
$$30 = |H| > 20 + 24 = 44$$

which is impossible.

6. $\underbrace{[T, R']}_{(1\ 2)}\ TR^{-2}\sigma_2 R^2\ (R^{-1}TR^{-1}TRTR)\ \sigma_2\ (R^{-1}TR^{-1}TRTR)$



7 (a)



$\rho = (3\ 7)$

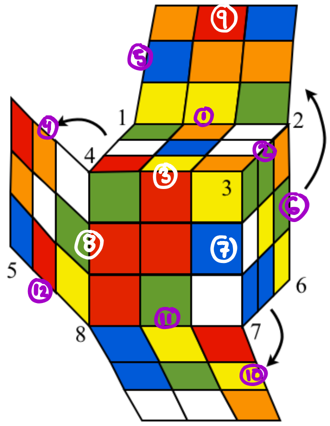
$\sigma = (1\ 2)(3\ 7)$

$\vec{v} = (0,0,1,0,0,0,2,0)$

$\vec{\omega} = (0,1,1,0,0,0,1,0,0,0,0,0)$

$\text{sign}(\rho) \neq \text{sign}(\sigma)$, as well as $\sum w_i \equiv 1 \pmod 2$, hence the configuration is not soluable.

(b)



$\rho = (1\ 5\ 2\ 6\ 3\ 8\ 4\ 7)$

$\sigma = (1\ 5\ 4\ 7\ 3\ 12\ 2\ 10\ 11\ 8\ 9\ 6)$

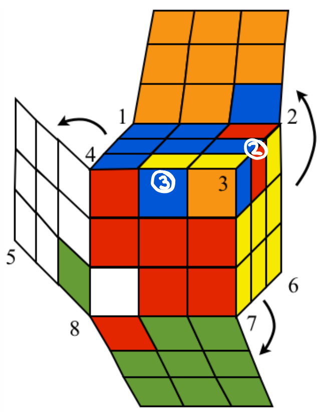
$\vec{v} = (1,1,0,1,2,2,0,2)$

$\vec{\omega} = (1,1,0,1,0,1,0,0,0,1,0,1)$

- $\rho$ & $\sigma$ are both odd
- $\sum v_i = 9 \equiv 0 \pmod 3$
- $\sum w_i = 6 \equiv 0 \pmod 2$

Therefore configuration is soluable.

8. (a)



$\rho = (2\ 3)$

$\sigma = (2\ 3)$

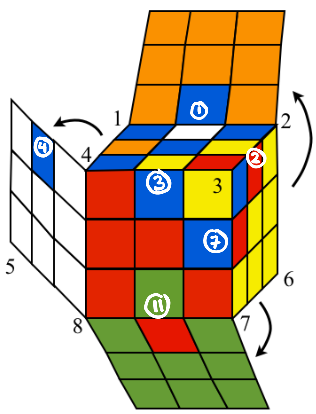
$\vec{v} = (0,2,2,0,0,0,0,2)$

$\vec{\omega} = (0,0,1,0,0,0,0,0,0,0,0,0)$

Since $\sum w_i \equiv 1 \pmod 2$ the configuration is not soluable. The quickest way to fix the cube is to remove an edge cubie, flip it over, and return it to the position it was removed from. This leaves $\rho, \sigma, \vec{v}$ unchanged, but changes $\sum w_i$ by 1 so $\sum w_i \equiv 0 \pmod 2$.

The coset to which this configuration belongs: $X_{1,0,1}$

(b)



$\left.\begin{array}{l}\rho = \varepsilon \\ \sigma = (1\ 4)(2\ 3\ 7)\end{array}\right\}$ $\text{sign}(\rho) \neq \text{sign}(\sigma)$

$\vec{v} = (0,0,2,0,0,0,0,0)$ $\rightarrow \sum v_i \equiv 2 \pmod 3$

$\vec{\omega} = (1,1,1,1,0,0,0,0,0,0,1,0)$

$\hookrightarrow \sum w_i \equiv 1 \pmod 2$

Most direct way to fix this is to do all of the following:

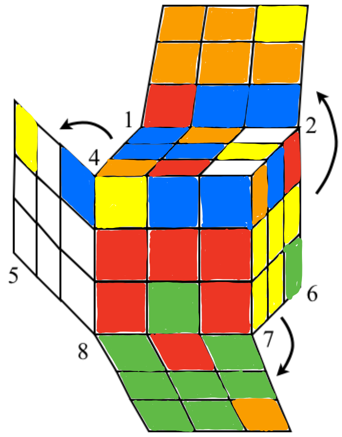
- swap two corner cubies (or two edge cubies),
- twist one corner cubie counterclockwise
- flip one edge cubie

The coset to which this configuration belongs is:

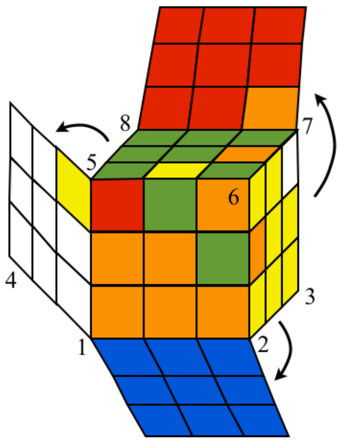
$X_{-1,2,1}$

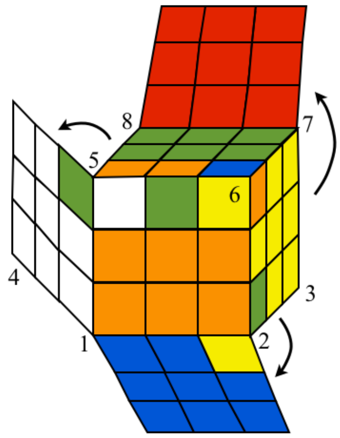
9

$\rho = (1\ 3)(2\ 4)$
$\sigma = \varepsilon$
$\vec{v} = (1,1,0,2,0,2,0,0)$
$\vec{w} = (1,1,1,0,0,0,0,0,0,0,1,0)$




10.




$\rho = (5\ 7)$
$\sigma = (6\ 10\ 9)$
$\vec{v} = (0,0,0,0,0,0,0,0)$
$\vec{w} = (0,0,0,0,0,1,0,0,1,1,0,0)$

$\rho = (2\ 6)$
$\sigma = \varepsilon$
$\vec{v} = (0,0,0,0,1,1,0,0)$
$\vec{w} = (0,0,0,0,0,0,0,0,1,0,0,0)$

Since $\sum v_i$ is different for each configuration then these two are not equivalent under cube moves.

11. Let $X = (\rho, \sigma, \vec{v}, \vec{w})$ be a configuration obtained from the solved state using only commutators. Since a commutator $[a,b]$ must simultaneously produce an even permutation on the corners and an even permutation on the edges, then

$$\text{sign}(\rho) = \text{sign}(\sigma) = 1.$$

This means commutators cannot solve configurations where $\rho$ & $\sigma$ are odd.

Therefore, it is not just Y & Z commutators are not enough to solve any configuration, commutators in general are not enough.

However, commutators are enough to get the puzzle to, at worst, a quarter turn from being solved. In this sense Singmaster was "almost" correct.