Textbook Reading: Chapters 17, 18, 19, 20 **Due Date:** Friday, April 9, 2021 by 11:59pm

Questions - Part 1

1. **Conjugation is an equivalence relation.** Let G be a group, show that the relation

$$g\mathcal{R}h \quad \Longleftrightarrow \quad g \text{ is a conjugate of } h,$$

is an equivalence relation.

2. Consider the group $\mathbb{Z}_{12}$ and the subgroup $H = \langle 4 \rangle = \{0, 4, 8\}$.
   (a) Are the following pairs of elements related under $\sim_H$?

   (i) $3, 7$ (ii) $5, 11$ (iii) $6, 9$

   (b) Find all (left) cosets of H in $\mathbb{Z}_{12}$.

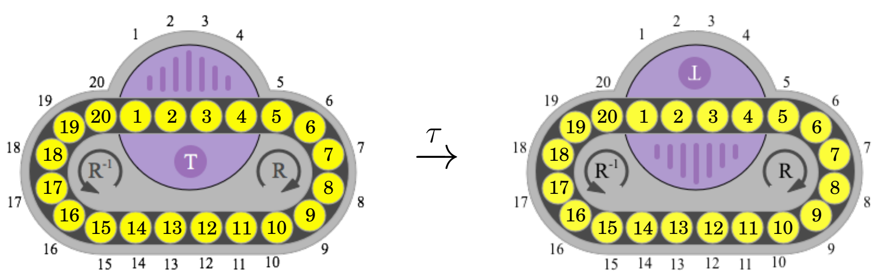
3. In S_7, are the following pairs of elements related under $\sim_H$ where $H = A_7$?

   (a) $(2\ 3)(4\ 6),\quad (1\ 3\ 5\ 7\ 4)$. (b) $(1\ 3\ 7\ 2),\quad (2\ 4\ 3\ 6\ 5)$.

4. Let H be a subgroup of a group G with $|H| = \frac{1}{2}|G|$.
   (a) Show that $a \notin H$ implies $G = H \cup aH$.
   (b) Show that $a \notin H$ implies $a^n H \neq a^{n+1} H$.
   (c) Deduce that H contains every element in G of odd order.

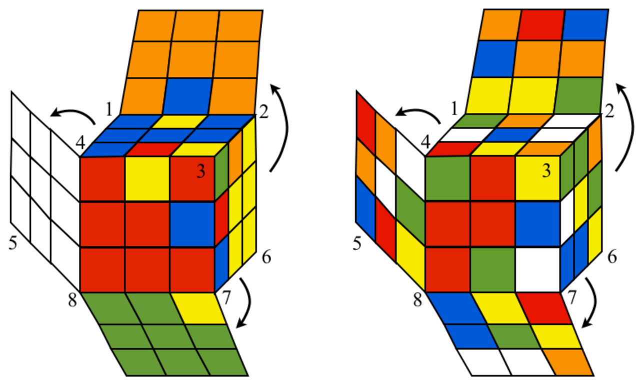
5. (a) How many 3-cycles are there in A_5?
   (b) How many 5-cycles are there in A_5?
   (c) Use Exercise 4 to show that A_5 has no subgroup of order 30.

6. **Back to Oval Track - Flipping the Turntable:** Find a move sequence τ that flips the turntable over but is the identity permutation on the disks (see figure). This shows that the orientation of the turntable is independent of the permutation of the disks.
(Play with the applet on the course website to help find a move sequence.)



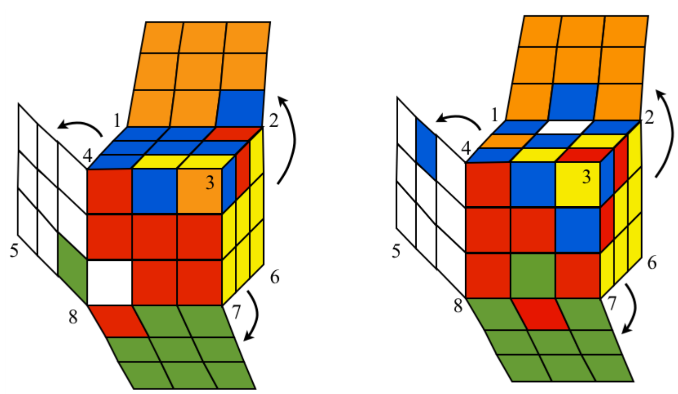
Questions - Part 2

7. For each of the following configurations (i) determine the position vector $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w}) \in S_8 \times S_{12} \times \mathbb{Z}_3^8 \times \mathbb{Z}_2^{12}$, and (ii) using the Fundamental Theorem of Cubology, determine whether it is a legal (i.e. solvable) configuration. (The corner cubicles are labeled, see Figure 20.2 in Chapter 20 for a labeling of the edge cubicles.)



(a) (b)

8. **Fix the Impossible Configurations.** In each part below, a configuration is shown. Show that each configuration is impossible by showing its position vector doesn't satisfy at least one of the three conditions of the First Fundamental Theorem of Cubology. Determine the quickest way to disassemble/reassemble it so that it becomes solvable. That is, decide if you have to swap two pieces, or flip a single edge, or twist a corner, or a combination of these, etc., in order to make it solvable.
Also, determine the representative $X_{i,j,k}$ (from Figure 20.6 in Chapter 20) for the coset to which each configuration belongs.



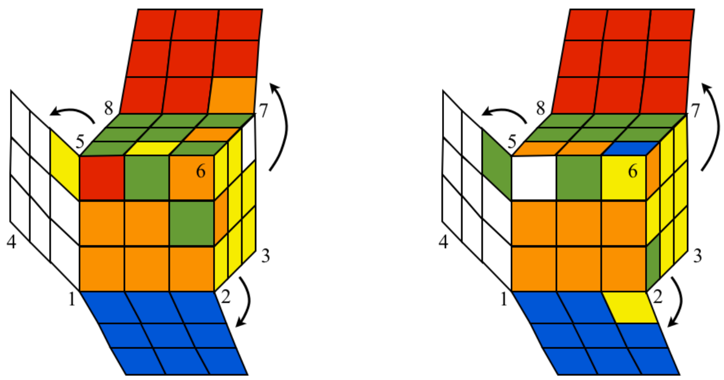
(a) (b)

9. For the following position vector $(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w}) \in S_8 \times S_{12} \times \mathbb{Z}_3^8 \times \mathbb{Z}_2^{12}$ draw the corresponding configuration. (Assume the standard orientation as shown in Figure 20.1 of Chapter 20.)

$$(\rho, \sigma, \boldsymbol{v}, \boldsymbol{w}) = ((2,4)(1,3), \varepsilon, (1,1,0,2,0,2,0,0), (1,1,1,0,0,0,0,0,0,0,1,0))$$

(Use the Rubik's cube puzzle template available on the assignments page of the course webpage.)

10. Are the following two (possibly illegal) configurations equivalent under cube moves? That is, by starting with a cube in configuration X can you get to configuration Y by just using Rubik's cube moves, or would you necessarily have to take the cube apart and reassemble it?



(a) X (b) Y

11. In Singmaster's book *Handbook of Cubik Math*, it is stated:

 > "The Z and Y commutators are so powerful that you could use either of them, applying each to different pairs of faces, and restore any scrambled cube without using any other [move sequences]."

 Unfortunately this statement is not true. Find the subtle reason why this statement is false, and provide an explanation. Provide a very small modification to this statement which will make it true.